THE COMBINATORIAL GAUSS DIAGRAM FORMULA FOR KONTSEVICH INTEGRAL

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ABSTRACT

In this paper, we shall give an explicit Gauss diagram formula for the Kontsevich integral of links up to degree four. This practical formula enables us to actually compute the Kontsevich integral in a combinatorial way.

Keywords: Kontsevich Integral, Gauss Diagram, Combinatorial, Vassiliev Invariant

1. Introduction

There are several types of formulas for Vassiliev invariants. However most of them are not suited for actual computations. So we provide more practical formulas for them.

Kontsevich [5] defined the famous link invariant (Kontsevich integral) using iterated integrals. The Kontsevich integral is a universal Vassiliev invariant of links which dominates all the other Vassiliev invariants. We give an explicit Gauss diagram formula for the Kontsevich integral up to degree four which is useful for actual computations.

In this paper we shall show the following results. We prove that the Kontsevich integral of links up to degree four can be expressed by some link invariants $v_1, v_2, v_{3.1}, v_{3.2}, v_{4.1}, v_{4.2}, v_{4.3}, v_{4.4}$ (See Theorem 1). We give an explicit Gauss diagram formula for these link invariants $v_1, v_2, v_{3.1}, v_{3.2}, v_{4.1}, v_{4.2}, v_{4.3}, v_{4.4}$ in terms of Gauss diagrams (See Theorem 2). This formula is obtained by evaluating Kontsevich integral using inductive argument. As a corollary, we obtain an explicit Gauss diagram formula for the power series expansion of the Homfly polynomial up to degree four, since the Kontsevich integral and the weight system of $su(N)$ gives the Homfly polynomial (See Corollary 1).

Witten [11] showed that the Chern-Simons quantum field theory gives a link invariant. We believe that the theory in this paper is the mathematical counter part of [2],[3],[4],[6], in which the quantum field theoretical method is used. In fact, these Gauss diagram formula for $v_2, v_{3.1}, v_{3.2}, v_{4.1}$ and $v_{4.2}$ coincide with those obtained by different methods in [2, 6], [3, 6], [4], [6] and [6] respectively. The Gauss diagram formulas for $v_{4.3}$ and $v_{4.4}$ are completely new.

The present paper is organized as follows. In section 2 we review the Kontsevich integral and discuss its property. In section 3 we give the Gauss diagram formula. In section 4 we discuss the relation to Homfly polynomial and give an example of the Gauss diagram formula. In section 5 we derive the Gauss diagram formula from Kontsevich integral. In section 6 we make a consistency check for the Gauss diagram formula.

2. Kontsevich Integral

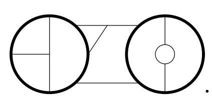
2.1. *Weight system*

We shall review the weight system as in [8] and fix our notation. (see also [1],[5],[9])

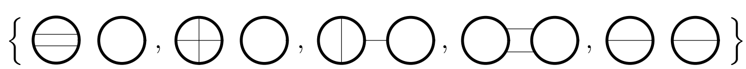
Definition 2.1. (CC Diagram) A *uni-trivalent graph* is a graph every vertex of which is either univalent or trivalent. A uni-trivalent graph is said to be vertex-oriented if at each trivalent vertex a cyclic order of edges is specified. Let $X = \cup_{i=1}^{n} S_i^1$ be n-oriented circles and G a vertex-oriented uni-trivalent graph. *A Chinese Character Diagram* (CC Diagram) is the pair $\{X, G\}$ where all the univalent vertices of G are on X. In all figures in the sequel, the component of X will be drawed by thick circles and the edges of graph G by thin lines. By convention, we set the orientation of each component of X and the orientation of each trivalent vertex counterclockwise, unless otherwise stated.

Two CC diagrams $D = \{X, G\}$, $D' = \{X', G'\}$ are regarded as equal if there is a homeomorphism $F : D \to D'$ such that $F|_X$ is a homeomorphism from X to X' which preserves orientation and $F|_G$ preserves the vertex orientation at each trivalent vertices. The degree of a CC diagram is defined to be half the number of vertices of the CC diagram.

For example, one of CC diagrams of degree 7 is



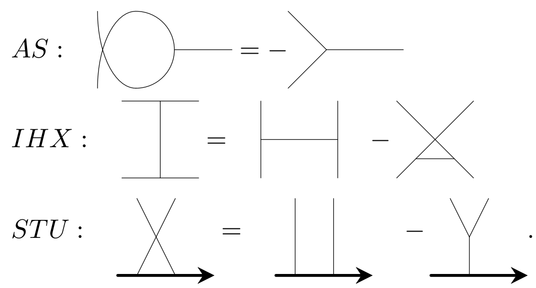
Definition 2.2. (Chord Diagram) A CC diagram is called a *Chord Diagram* if all the vertices are univalent. An edge of a chord diagram is called a *chord*. Then it is clear that the degree of a chord diagram is equal to the number of the chords. We give an example of chord diagrams of degree 2 with 2-circles :



Definition 2.3. Let $\mathfrak{D}^t$ be the set of all CC diagrams. We define the vector space $\mathcal{A}$ by

$$\mathcal{A} = \mathrm{span}(\mathfrak{D}^t)/\mathrm{AS,IHX,STU}$$

where the AS, IHX and STU relations are shown below:



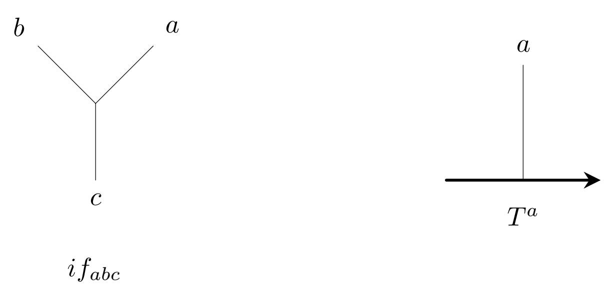
□

Although we restrict our consideration to the natural (fundamental) representation of $su(N)$, all the argument in the sequel is valid for any simple Lie algebra and its irreducible representation. The matrix basis $\{T^a\}_{a\in I}$ of the natural (fundamental) representation of $su(N)$ are normalized as follows:

$$[T^a, T^b] = if^{abc}T^c, \quad \mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$$

with the structure constant if^{abc}.

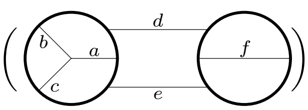
Definition 2.4. (Weight System) We define a map $W_{su(N)} : \mathcal{A} \to \mathbb{C}$ which is called the weight system for the natural (fundamental) representaion of $su(N)$. Let $D = \{X, G\}$ be a CC diagram and $E(G)$ the set of all the edges of the graph G. By a labelling of D, we mean a map $\rho : E(G) \to I$. For each labelling, we assign the structure constant if^{abc} to each trivalent vertex where the three edges around the vertex are labeled by a, b, c along its orientation. We assign the basis T^a to each univalent vertex where the edge emanating from the vertex is labeled by a.



Define $W_{su(N)}(D)$ as follows. For each labelling, make the product of all the assigned structure constants if^{abc} and all the traces of the product of the basis T^a along each circle of X. Define $W_{su(N)}(D)$ to be the sum of these products where the sum is taken over all the labelling:

$$W_{su(N)}(D) = \frac{x^m}{N^n} \sum_{a,b,c,\cdots=1}^{N^2-1} \{\text{product of } (if^{abc})\} \prod^{n} \{\text{Trace}(\text{product of } T^a)\},$$

where m denotes the degree of D and n the number of the circles. For example,

$W_{su(N)}\Bigg($ 

$\Bigg)$

$$= \frac{x^5}{N^2} \sum_{a,b,c,d,e,f=1}^{N^2-1} if^{abc}\text{Tr}(T^eT^cT^bT^dT^a)\text{Tr}(T^fT^eT^fT^d).$$

2.2. *Kontsevich integral*

Definition 2.5. (Kontsevich Integral) Let $\hat{\mathcal{A}}$ be the quotient of $\mathcal{A}$ by the framing independence relation. We shall define the Kontsevich integral on $\hat{\mathcal{A}}$. For more detail, see [1],[5]. Let $X = \cup_{i=1}^n S_i^1$ be n-oriented circles and $\vec{x} : X \to \mathbb{R}^3$ an imbedding. An n-component oriented link $\mathbf{L}$ is its image $\mathbf{L} = \{\mathbf{K}_1, \cdots, \mathbf{K}_n\}$ ($\mathbf{K}_i = \vec{x}(S_i^1)$) with the natural orientation. Let us introduce coordinates z, t in $\mathbb{R}^3$ by $z = x_1 + ix_2 \in \mathbb{C}$, $t = x_3 \in \mathbb{R}$. Let $t_{\min}$ (resp. $t_{\max}$) be the minimum (resp. maximum) value of t on $\mathbf{L}$. We consider m-planes $t = t_k, (k = 1, \cdots, m)$ where $t_{\min} < t_1 < \cdots < t_m < t_{\max}$. Define a height function π on X by $\pi(s) = t(\vec{x}(s))$, $(s \in X)$. The inverse function $\pi^{-1}(t)$ is a multi-valued function on $\mathbb{R}$. So set $(\pi^{-1})(t_k) = \{s_k^1, \cdots, s_k^{n(t_k)}\}$, where $n(t_k)$ denotes the number of points on the section $t = t_k$ of the link $\mathbf{L}$. For $1 \le i \le j \le n(t_k)$, set $z_{ij}(t_k) = z\big\{\vec{x}(s_k^i) - \vec{x}(s_k^j)\big\}$. Define the collection of all the pairings by $P = \{(i_1, j_1), (i_2, j_2), \cdots, (i_m, j_m) : 1 \le i_k < j_k \le n(t_k) \quad (k = 1, \cdots, m)\}$. For a pairing $p \in P$, write D_p for the chord diagram of degree m obtained by joining $s_k^{i_k}$ and $s_k^{j_k}$ by chords on X $(k = 1, \cdots, m)$. It is to be regarded as an element of the quotient $\hat{\mathcal{A}}$. The Kontsevich integral is defined as follows:

$$Z(\mathbf{L}) = \sum_{m=0}^{\infty} \frac{1}{(i\pi)^m} \int_{t_{\max} > t_1 > \cdots > t_m > t_{\min}} \sum_{p \in P} D_p \prod_{k=1}^{m} \{\epsilon \; d\log(z_{i_kj_k}(t_k))\}, \qquad (2.1)$$

where the signature ϵ in front of $d\log(z_{i_kj_k}(t_k))$ is $+1$ if the two orientations of $\mathbf{L}$ at $\vec{x}(s_k^{i_k})$ and $\vec{x}(s_k^{j_k})$ are the same with respect to t-axis and -1 if they are different. Notice we have used the slightly different normalization from [1],[5].

Definition 2.6. (Modified Kontsevich Integral) Define $Z_W(\mathbf{L})$ by

$$Z_W(\mathbf{L}) = \hat{W}_{su(N)}(Z(\mathbf{L})), \qquad (2.2)$$

where $\hat{W}_{su(N)}$ is the renormalized version of $W_{su(N)}$ to be compatible with the framing independence (see [1] page 426). It is known that the Kontsevich integral is invariant under only horizontal deformation of $\mathbf{L}$. So we define the modified Kontsevich integral by

$$\hat{Z}_W(\mathbf{L}) = Z_W(\mathbf{L}) Z_W(U_0)^{-m(\mathbf{L})}, \tag{2.3}$$

where $m(\mathbf{L})$ denotes the number of maximal points of link $\mathbf{L}$ and U_0 is a knot given in Figure 1. It is known that $\hat{Z}_W(\mathbf{L})$ is invariant under arbitrary deformations of the link $\mathbf{L}$. We remark $\hat{Z}_W(\mathbf{L})$ is a formal power series with respect to x.

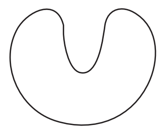


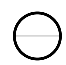
Fig. 1. U_0

2.3. *The Kontsevich integral up to degree four*

In this section, we shall prove that the modified Kontsevich integral of links up to degree four can be expressed by some link invariants $v_1, v_2, v_{3.1}, v_{3.2}, v_{4.1}, v_{4.2}, v_{4.3}$ and $v_{4.4}$ (Theorem 1).

Definition 2.7. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series of x. Define $\big[f(x)\big]^{(k)}$ by

$$\big[f(x)\big]^{(k)} = \sum_{n=0}^{k} a_n x^n.$$

Definition 2.8. Let $\mathbf{L}$ be a link and D a chord diagram of degree m without any isolated chord (namely, D cannot be decomposed as the product of  and a chord diagram). Define $\left\langle\!\!\left\langle \mathbf{L}, D \right\rangle\!\!\right\rangle$ by

$$\left\langle\!\!\left\langle \mathbf{L}, D \right\rangle\!\!\right\rangle = \frac{1}{(i\pi)^m} \int_{t_{\max} > t_1 > \cdots > t_m > t_{\min}} \sum_{p \in P} \prod_{k=1}^{m} \big\{\epsilon \; d\log(z_{i_k j_k}(t_k))\big\} \Theta(D_p, D),$$

where D_p denotes the chord diagram corresponding to the pairing $p \in P$. The sum is taken over all the pairings $p \in P$. $\Theta(D_p, D)$ is defined by

$$\Theta(D_p, D) = \begin{cases} 1 & \text{if } D_p = D \\ 0 & \text{if } D_p \neq D \end{cases}.$$

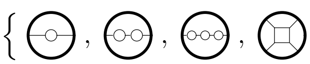
More generally, for a formal linear combinaiton of chord diagrams $\sum_i b_i D_i$ ($b_i \in \mathbb{C}$, D_i is a chord diagram without any isolated chord), set

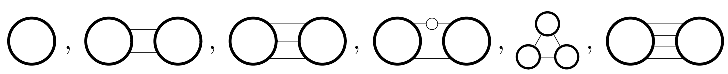
$$\left\langle\!\!\left\langle \mathbf{L}, \sum_i b_i D_i \right\rangle\!\!\right\rangle = \sum_i b_i \left\langle\!\!\left\langle \mathbf{L}, D_i \right\rangle\!\!\right\rangle.$$

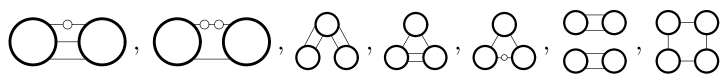
Theorem 1. Let $\mathbf{L} = \{\mathbf{K}_1, \cdots, \mathbf{K}_n\}$ be a link where $\mathbf{K}_i$ denotes each component of the link $\mathbf{L}$ ($i = 1, \cdots, n$). The modified Kontsevich integral up to degree four $\left[\hat{Z}_W(\mathbf{L})\right]^{(4)}$ can be expressed as

$$\left[\hat{Z}_W(\mathbf{L})\right]^{(4)} = W^{(4)}_{su(N)}\left(\left\{\exp\Big(\sum_{D\in\mathfrak{D}_K} D\ w(D:\mathbf{L})\Big)\right\}\left\{\sum_{D\in\mathfrak{D}_L} D\ w(D:\mathbf{L})\right\}\right), \quad (2.4)$$

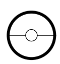
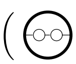
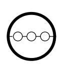
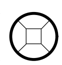
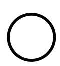
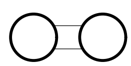
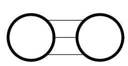
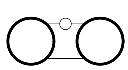
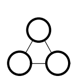
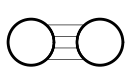
where $W^{(4)}_{su(N)}(D) = \left[W_{su(N)}(D)\right]^{(4)}$, the sum is taken over the following CC diagrams:

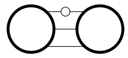
$\mathfrak{D}_K = \Big\{$  $\Big\}$,

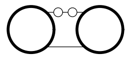
$\mathfrak{D}_L = \Big\{$ 

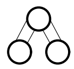
 $\Big\}$ (2.5)

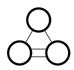
and

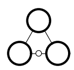
- $w\Big($  $:\mathbf{L}\Big) = (-\frac{1}{2})\sum_{i=1}^{n} v_2(\mathbf{K}_i)$, • $w\Big($  $:\mathbf{L}\Big) = (-\frac{1}{2})^2\sum_{i=1}^{n} v_{3.1}(\mathbf{K}_i)$,
- $w\Big($  $:\mathbf{L}\Big) = (-\frac{1}{2})^3\sum_{i=1}^{n} v_{4.1}(\mathbf{K}_i)$, • $w\Big($  $:\mathbf{L}\Big) = \sum_{i=1}^{n} v_{4.2}(\mathbf{K}_i)$,
- $w\Big($  $:\mathbf{L}\Big) = 1$, • $w\Big($  $:\mathbf{L}\Big) = \sum_{1\leq i<j\leq n} \frac{1}{2}\big(v_1(\{\mathbf{K}_i, \mathbf{K}_j\})\big)^2$,
- $w\Big($  $:\mathbf{L}\Big) = \sum_{1\leq i<j\leq n} \frac{1}{3!}\big(v_1(\{\mathbf{K}_i, \mathbf{K}_j\})\big)^3$,
- $w\Big($  $:\mathbf{L}\Big) = (-\frac{1}{2})\sum_{1\leq i<j\leq n} v_{3.2}(\{\mathbf{K}_i, \mathbf{K}_j\})$,
- $w\Big($  $:\mathbf{L}\Big) = \sum_{1\leq i<j<k\leq n} v_1(\{\mathbf{K}_i, \mathbf{K}_j\})v_1(\{\mathbf{K}_j, \mathbf{K}_k\})v_1(\{\mathbf{K}_k, \mathbf{K}_i\})$,
- $w\Big($  $:\mathbf{L}\Big) = \sum_{1\leq i<j\leq n} \frac{1}{4!}\big(v_1(\{\mathbf{K}_i, \mathbf{K}_j\})\big)^4$,

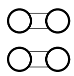
- $w\Big($  $: \mathbf{L}\Big) = (-\frac{1}{2}) \sum_{1\le i<j\le n} \frac{1}{2} v_1(\{\mathbf{K}_i, \mathbf{K}_j\}) v_{3.2}(\{\mathbf{K}_i, \mathbf{K}_j\}),$

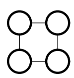
- $w\Big($  $: \mathbf{L}\Big) = (-\frac{1}{2})^2 \sum_{1\le i<j\le n} v_{4.3}(\{\mathbf{K}_i, \mathbf{K}_j\}),$

- $w\Big($  $: \mathbf{L}\Big) = \sum_{\substack{1\le i<j<k\le n\\ 1\le j<i<k\le n\\ 1\le j<k<i\le n}} \frac{1}{2}\big(v_1(\{\mathbf{K}_i, \mathbf{K}_j\})\big)^2 \frac{1}{2}\big(v_1(\{\mathbf{K}_i, \mathbf{K}_k\})\big)^2,$

- $w\Big($  $: \mathbf{L}\Big) = \sum_{\substack{1\le i<j<k\le n\\ 1\le j<i<k\le n\\ 1\le j<k<i\le n}} v_1(\{\mathbf{K}_i, \mathbf{K}_j\}) v_1(\{\mathbf{K}_i, \mathbf{K}_k\}) \frac{1}{2}\big(v_1(\{\mathbf{K}_j, \mathbf{K}_k\})\big)^2,$

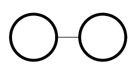
- $w\Big($  $: \mathbf{L}\Big) = (-\frac{1}{2}) \sum_{1\le i<j<k\le n} v_{4.4}(\{\mathbf{K}_i, \mathbf{K}_j, \mathbf{K}_k\}),$

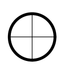
- $w\Big($  $: \mathbf{L}\Big) = \sum_{\substack{1\le i<j<k<l\le n\\ 1\le i<k<j<l\le n\\ 1\le i<k<l<j\le n}} \frac{1}{2}(v_1(\{\mathbf{K}_i, \mathbf{K}_j\}))^2 \frac{1}{2}(v_1(\{\mathbf{K}_k, \mathbf{K}_l\}))^2,$

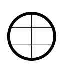
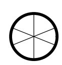
- $w\Big($  $: \mathbf{L}\Big)$

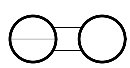
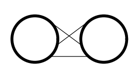
$$= \sum_{\substack{1\le i<k<j<l\le n\\ 1\le i<j<k<l\le n\\ 1\le i<j<l<k\le n}} v_1(\{\mathbf{K}_i, \mathbf{K}_j\}) v_1(\{\mathbf{K}_j, \mathbf{K}_k\}) v_1(\{\mathbf{K}_k, \mathbf{K}_l\}) v_1(\{\mathbf{K}_l, \mathbf{K}_i\}), \tag{2.6}$$

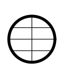
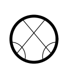
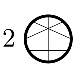
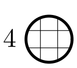
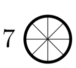
and $v_1, v_2, v_{3.1}, v_{3.2}, v_{4.1}, v_{4.2}, v_{4.3}, v_{4.4}$ are given as follows:

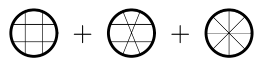
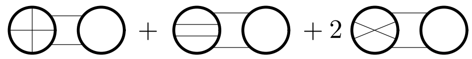
- $v_1(\{\mathbf{K}_i, \mathbf{K}_j\}) = \Big\langle\!\!\Big\langle \{\mathbf{K}_i, \mathbf{K}_j\},$  $\Big\rangle\!\!\Big\rangle,$ (2.7)

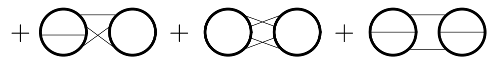
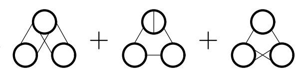
- $v_2(\mathbf{K}_i) = \Big\langle\!\!\Big\langle \mathbf{K}_i,$  $\Big\rangle\!\!\Big\rangle - \frac{1}{6} m(\mathbf{K}_i),$ (2.8)

- $v_{3.1}(\mathbf{K}_i) = \Big\langle\!\!\Big\langle \mathbf{K}_i,$  $+ 2$  $\Big\rangle\!\!\Big\rangle,$ (2.9)

- $v_{3,2}(\{\mathbf{K}_i, \mathbf{K}_j\}) = \Big\langle\!\!\Big\langle \{\mathbf{K}_i, \mathbf{K}_j\},$  $+$  $\Big\rangle\!\!\Big\rangle,$ (2.10)

- $v_{4.1}(\mathbf{K}_i) = \Big\langle\!\!\Big\langle \mathbf{K}_i,$  $+$  $+ 2$  $+ 4$  $+ 5$  $+ 7$  $\Big\rangle\!\!\Big\rangle$ $+ \frac{1}{360} m(\mathbf{K}_i),$

- $v_{4.2}(\mathbf{K}_i) = \left\langle\!\!\left\langle \mathbf{K}_i,\right.\right.$  $\left.\left.\right\rangle\!\!\right\rangle - \dfrac{1}{360} m(\mathbf{K}_i),$
- $v_{4.3}(\{\mathbf{K}_i, \mathbf{K}_j\}) = \left\langle\!\!\left\langle \{\mathbf{K}_i, \mathbf{K}_j\},\right.\right.$ 

   $\left.\left.\right\rangle\!\!\right\rangle,$
- $v_{4.4}(\{\mathbf{K}_i, \mathbf{K}_j, \mathbf{K}_k\}) = \left\langle\!\!\left\langle \{\mathbf{K}_i, \mathbf{K}_j, \mathbf{K}_k\},\right.\right.$  $\left.\left.\right\rangle\!\!\right\rangle,$ (2.11)

where $m(\mathbf{K}_i)$ is the number of maximal points of $\mathbf{K}_i$.

Moreover $v_1, v_2, v_{3.1}, v_{3.2}, v_{4.1}, v_{4.2}, v_{4.3}, v_{4.4}$ are link invariants.

Proof of Theorem 1. The computation is long but straightforward. Kontsevich integral (2.1) can be rewritten in the following form:

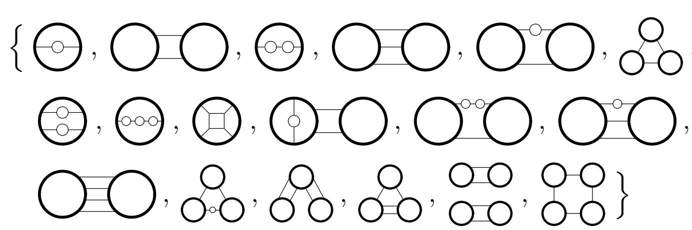
$$Z(\mathbf{L}) = \sum_{m=0}^{\infty} \sum_{D \in \mathfrak{D}_m} D \left\langle\!\!\left\langle \mathbf{L}, D \right\rangle\!\!\right\rangle,$$

where $\mathfrak{D}_m$ denotes the set of all chord diagrams of degree m which have just n circles and no isolated chord. From (2.2), we have

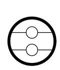
$$\left[Z_W(\mathbf{L})\right]^{(4)} = \hat{W}_{su(N)}\left(\sum_{m=0}^{4} \sum_{D \in \bar{\mathfrak{D}}_m} D \left\langle\!\!\left\langle \mathbf{L}, D \right\rangle\!\!\right\rangle \right), \tag{2.12}$$

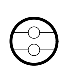
where $\bar{\mathfrak{D}}_m = \{D \in \mathfrak{D}_m \mid \hat{W}_{su(N)}(D) \neq 0\}$ and we give the table of $\bar{\mathfrak{D}}_m$ in Appendix A.

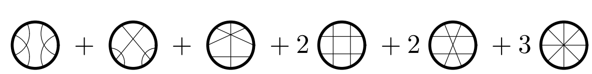
In (2.12), we express each chord diagram D in front of $\left\langle\!\!\left\langle \mathbf{L}, D \right\rangle\!\!\right\rangle$ as a linear combination of the following CC diagrams

 (2.13)

regarded as an element in $\hat{\mathcal{A}}$ using Appendix B.

Next, we compute each cofficient of the CC diagram in (2.13). For example, the cofficient of  is

$\Big($the cofficient of $(-\dfrac{1}{2})^2$  $\Big)$

$$= \sum_{i=1}^{n} \left\langle\!\!\left\langle \mathbf{K}_i, \right.\right.$$  $\left.\left.\right\rangle\!\!\right\rangle$

$$
\begin{aligned}
&+\sum_{i<j}\left\langle\!\!\left\langle \{\mathbf{K}_i,\mathbf{K}_j\},\{\bigoplus\ \bigoplus\}\right\rangle\!\!\right\rangle\\
=\ & \sum_{i=1}^{n}\frac{1}{2}\left\langle\!\!\left\langle \mathbf{K}_i,\bigoplus\right\rangle\!\!\right\rangle^2+\sum_{i<j}\left\langle\!\!\left\langle \mathbf{K}_i,\bigoplus\right\rangle\!\!\right\rangle\left\langle\!\!\left\langle \mathbf{K}_j,\bigoplus\right\rangle\!\!\right\rangle\\
=\ & \frac{1}{2}\left\{\sum_{i=1}^{n}\left\langle\!\!\left\langle \mathbf{K}_i,\bigoplus\right\rangle\!\!\right\rangle\right\}^2.
\end{aligned}
$$

See Appendix C for the other cofficients of the CC diagrams in (2.13). Inserting these result into (2.3) and using

$$
\begin{aligned}
&\left[Z_W(U_0)^{-1}\right]^{(4)}\\
&\quad = W^{(4)}_{su(N)}\left(\exp\left\{(-\frac{1}{2})\ (-\frac{1}{6})+(-\frac{1}{2})^3\ \frac{1}{360}+\ (-\frac{1}{360})\right\}\right),
\end{aligned}
$$

we have (2.4).

Next we prove the invariace of $v_1,v_2,v_{3.1},v_{3.2},v_{4.1},v_{4.2},v_{4.3},v_{4.4}$. Let $\mathbf{L}$ be a 1-component link (that is a knot). Then $v_1,v_{3.2},v_{4.3},v_{4.4}$ in (2.4) vanish since they are defined for more than 2-component link. Since $\hat{Z}_W(L)$ is a link invariant and

$$
\begin{aligned}
&W_{su(N)}\left((-\frac{1}{2})\ \right),\quad W_{su(N)}\left((-\frac{1}{2})^2\ \right),\\
&W_{su(N)}\left((-\frac{1}{2})^3\ \right),\quad W_{su(N)}\left(\ \right),
\end{aligned}
\tag{2.14}
$$

are linearly independent as polynomials of x,N, we see that $v_2,v_{3.1},v_{4.1},v_{4.2}$ are link invariant. We can also prove the invariance of $v_1,v_{3.2},v_{4.3},v_{4.4}$ in the same way. $\Box$

3. Gauss Diagram Formula

In this section, we shall give an explicit Gauss diagram formula for the link invariants v_1, v_2, $v_{3.1}$, $v_{3.2}$, $v_{4.1}$, $v_{4.2}$, $v_{4.3}$, $v_{4.4}$ in terms of Gauss diagrams (See Theorem 2). Before we state Theorem 2, we shall fix the notation for this purpose.

3.1. *Pairing* $\langle\hat{G},\hat{D}\rangle_\chi$

Definition 3.1. (Link Diagram) Let $X=\cup_{i=1}^n S_i^1$ be n-oriented circles and $\vec{y}: X\to\mathbb{R}^2$ an immersion. An n-component *oriented link diagram* L is its image $L=\{K_1,\cdots,K_n\}$ ($K_i=\vec{y}(S_i^1)$) together with the information of overpass or underpass at each crossing. We write the information of each crossing as in Figure 2. We call ±1 assigned to a crossing *the signature of the crossing*. We often write $\pm$ instead of ±1 for the signature of the crossing.

Definition 3.2. (IL Diagram) Let D be a chord diagram and $C(D)$ the set of all chords of D. By an integer-labelling of D, we mean a map $\kappa: C(D)\to\mathbf{Z}$. *An*

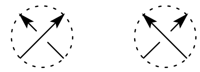


Fig. 2. the information of the crossing

Integer-Labeled Chord Diagram (IL Diagram) is a pair $\{D, \kappa\}$ of a chord diagram D together with an integer-labelling κ. Two IL diagram $\{D, \kappa\}$, $\{D', \kappa'\}$ are regarded as equal if D, D' are equal as chord diagrams and the homeomorphism $F : D \to D'$ preserves integer-labelling $\kappa'(F(c)) = \kappa(c)$ $(c \in C(D))$. $\square$

We shall define a *Gauss Diagram* and *ML Diagram* as special cases of IL Diagrams.

Definition 3.3. (Gauss Diagram) An IL diagram $\{G, \epsilon\}$ is called a *Gauss Diagram* if $\epsilon(c) = \pm 1$ $(c \in C(G))$. An integer-labelling ϵ of the Gauss diagram is called a *signature-labelling.*

Let $\{L : a_1, \cdots, a_m\}$ be a link diagram L where we select some distinct crossings $a_1, \cdots, a_m$ out of all crossings of L. Define a Gauss diagram $P(\{L : a_1, \cdots, a_m\})$ as follows. For each a_i, set $\vec{y}^{-1}(a_i) = \{s(a_i), s'(a_i)\}$ as the inverse image of a_i. For each crossing a_i, we join $s(a_i), s'(a_i)$ by a chord on X and label this chord by the signature of a_i $(i = 1, \cdots, m)$. We define a Gauss diagram $P(\{L : a_1, \cdots, a_m\})$ to be the result.

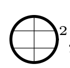
Specially, If $\{a_1, \cdots, a_m\}$ are all the crossings of L (this means we select all the crossings of L) , we write $G(L) = P(\{L : a_1, \cdots, a_m\})$ and call it the Gauss diagram of L.

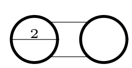
For example, see Figure 3.

Definition 3.4. (ML Diagram) An IL diagram $\{D, m\}$ is called a *Multiplicity-Labeled Diagram* (ML Diagram) if $m(c) = 1, 2$ $(c \in C(D))$. In figures, we draw a chord c with $m(c) = 1$ by a thin line and a chord c with $m(c) = 2$ by a thin line with a letter "2" as follows:

$$\overline{\qquad\qquad}\ \text{m(c)=1,} \qquad\qquad \overset{2}{\overline{\qquad\qquad}}\ \text{m(c)=2.}$$

We give two examples of ML diagrams,



, 

.

Definition 3.5. Let $\hat{G} = \{G, \epsilon\}$ be a Gauss diagram and $\hat{D} = \{D, m\}$ a ML diagram. Let $\psi : D \to G$ be an embedding of D into G which maps the circles of D to those of G preserving the orientations and each chord of D to a chord of G. Let $C(G)$ be the set of all chords of G. For ψ, define a map $\kappa_\psi : C(G) \to \{0, 1, 2\}$ by

$$\kappa_\psi(c) = \begin{cases} m(\psi^{-1}(c)) & \text{if } \ c \in \psi(D) \\ 0 & \text{if } \ c \notin \psi(D) \end{cases}$$

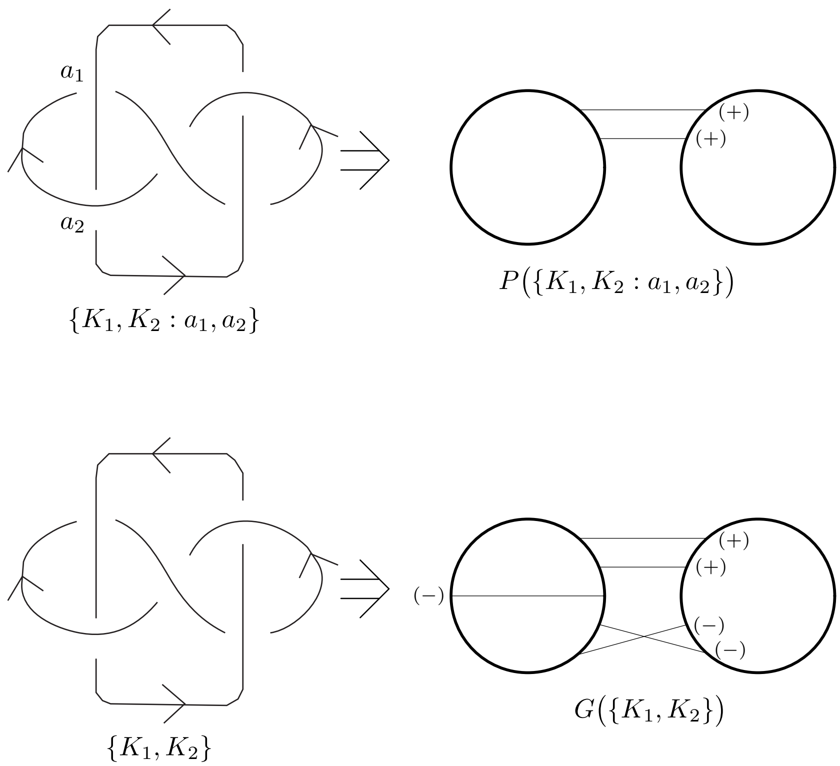


Fig. 3.

Two embedding ψ, φ are said to be equal if $\kappa_\psi = \kappa_\varphi$. The equivalence class of an embedding ψ is denoted by $[\psi]$.

Let $C(D)$ be the set of all chords of D. Define $\mathcal{E}([\psi])$ by

$$\mathcal{E}([\psi]) = \prod_{c \in C(D)} \big\{\epsilon(\psi(c))\big\}^{m(c)},$$

where the product is taken over all chords of D. Notice this definition is well defined.

Define a pairing of a Gauss diagram and ML diagram $\langle \hat{G}, \hat{D} \rangle_\chi$ by

$$\langle \hat{G}, \hat{D} \rangle_\chi = \sum_{[\psi]} \mathcal{E}([\psi]),$$

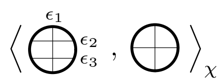
where the sum is taken over all the distinct equivalence classes $[\psi]$.

Let $\hat{G}_i$ be a Gauss diagram and $\hat{D}_i$ a ML diagram. More generally, for formal

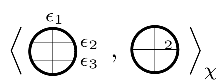
linear combinations $\sum_i b_i\ \hat{G}_i$ and $\sum_j c_j\ \hat{D}_j$ $(b_i, c_j \in \mathbf{C})$, set

$$\Big\langle \sum_i b_i\ \hat{G}_i, \sum_j c_j\ \hat{D}_j \Big\rangle_\chi = \sum_i \sum_j b_i\ c_j\ \langle \hat{G}_i, \hat{D}_j \rangle_\chi,$$

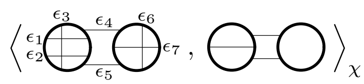
Example 3.6.



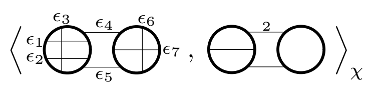
$$= \epsilon_1\epsilon_2 + \epsilon_1\epsilon_3,$$



$$= (\epsilon_1)^2\epsilon_2 + \epsilon_1(\epsilon_2)^2 + (\epsilon_1)^2\epsilon_3 + \epsilon_1(\epsilon_3)^2,$$



$$= (\epsilon_1 + \epsilon_2 + \epsilon_7)\epsilon_4\epsilon_5,$$



$$= (\epsilon_1 + \epsilon_2 + \epsilon_7)\big((\epsilon_4)^2\epsilon_5 + \epsilon_4(\epsilon_5)^2\big),$$

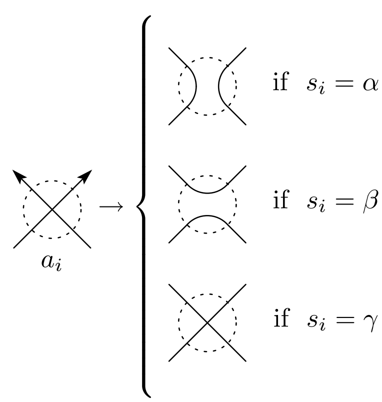
where $\epsilon_i = \pm 1$ denotes the signature-labelling.

3.2. *Gauss diagram formula*

Next, we shall introduce a concept for splitting the crossings of a link diagram.

Definition 3.7. (Splitting of the crossings) Let $\{L : a_1, \cdots, a_m\}$ be a link diagram L where we select some distinct crossings $a_1, \cdots, a_m$ out of all crossings of L. By a splitting information, we mean a finite sequence $[s_1, \cdots, s_m]$ $(s_i = \alpha, \beta, \gamma)$, where α, β, γ are formal letters. For example, $[\alpha, \beta, \alpha, \gamma]$ $(m = 4)$.

We shall define a link diagram $Q(\{L : a_1, \cdots, a_m\}, [s_1, \cdots, s_m])$ as follows. For $1 \leq i \leq m$, we replace each crossing a_i by



and give any orientation to the resulting diagram.

Define $Q(\{L : a_1, \cdots, a_m\}, [s_1, \cdots, s_m])$ to be the resulting oriented link diagram. We remark that our caluculation in the sequel does not depend on the paticular choice of the orientation of $Q(\{L : a_1, \cdots, a_m\}, [s_1, \cdots, s_m])$.

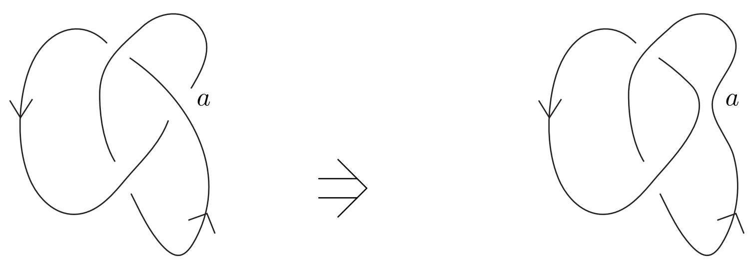
More generally, for a formal linear combination of splitting information $\sum_i b_i \ \delta_i$ ($b_i \in \mathbf{C}, \quad \delta_i = [s^i_1, \cdots, s^i_m]$), set

$$Q(\{L : a_1, \cdots, a_m\}, \sum_i b_i \ \delta_i) = \sum_i b_i \ Q(\{L : a_1, \cdots, a_m\}, \delta_i).$$

Example 3.8. We give a trivial example:

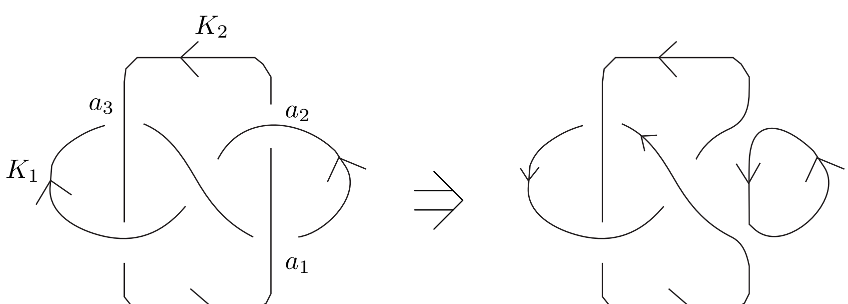
$$Q(\{L : a\}, [\gamma]) = L$$

Example 3.9. We give two nontrivial examples (see Figure 4 and Figure 5). As for Figure 5, notice the orientation of knots is partly changed.



$\{K : a\}$ $\qquad$ $Q(\{K : a\}, [\alpha])$

Fig. 4. $\{K : a\} \rightarrow Q(\{K : a\}, [\alpha])$



$\{K_1, K_2 : a_1, a_2, a_3\}$ $\qquad$ $Q(\{K_1, K_2 : a_1, a_2, a_3\}, [\beta, \beta, \gamma])$

Fig. 5. $\{K_1, K_2 : a_1, a_2, a_3\} \rightarrow Q(\{K_1, K_2 : a_1, a_2, a_3\}, [\beta, \beta, \gamma])$

Definition 3.10. Let $L = \{K_1, K_2, \cdots, K_n\}$ be an n-component link diagram. Define $S(L)$ to be the formal sum of each component K_i

$$S(L) = \sum_{i=1}^{n} K_i.$$

For example, see Figure 6.

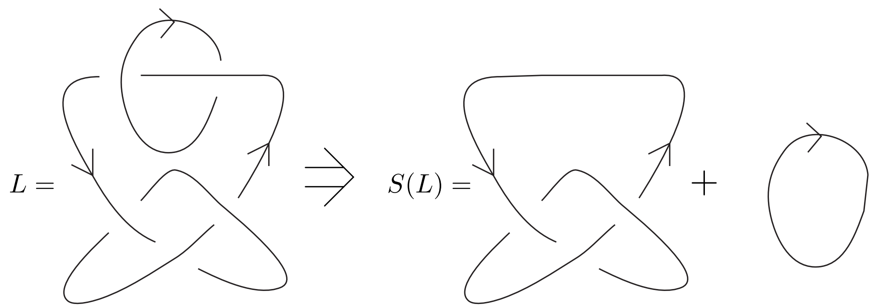


Fig. 6. $L = \{K_1, K_2\}$ and $S(L) = K_1 + K_2$

Definition 3.11. (L and $\alpha(L)$) Let L be an n-component link diagram. Let $\alpha(L)$ be a trivial link diagram of n-separated trivial knots which is obtained by swiching the signature of the crossings of L properly (see Figure 7). There are several ways to obtain $\alpha(L)$ from L. So $\alpha(L)$ cannot be uniquely determined from L. But the caluculation in the sequel does not depend on the way we choose.

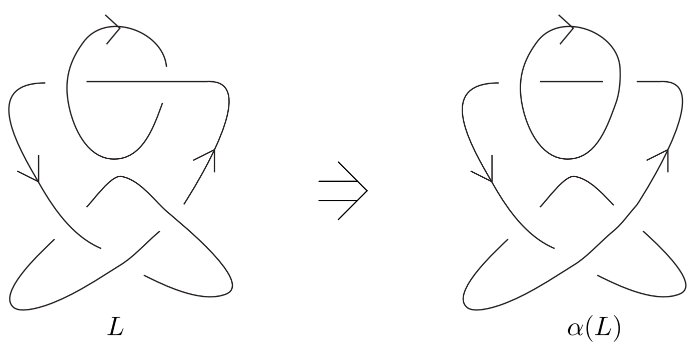
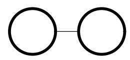
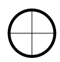
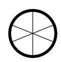
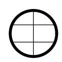
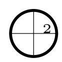
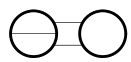
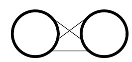
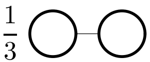
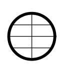
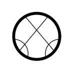
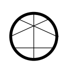
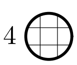
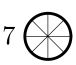
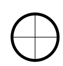
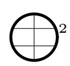
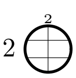
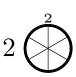
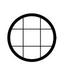
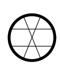
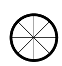
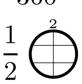
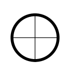
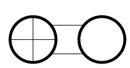
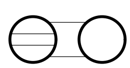
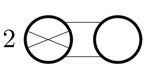
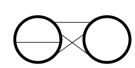
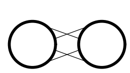
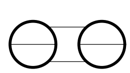
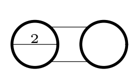
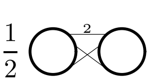


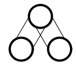
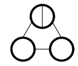
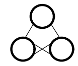
Fig. 7. $L \to \alpha(L)$

Definition 3.12. Let L_i be a link diagram. For a formal linear combination $\sum_i b_i L_i$ ($b_i \in \mathbf{C}$), we extend the definition of G, α, S by

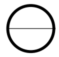
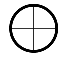
$$G(\sum_i b_i\ L_i) = \sum_i b_i\ G(L_i), \qquad \alpha(\sum_i b_i\ L_i) = \sum_i b_i\ \alpha(L_i),$$
$$S(\sum_i b_i\ L_i) = \sum_i b_i\ S(L_i).$$

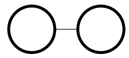
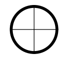
Theorem 2. (Gauss diagram formula) Let K, $\{K_1, K_2\}$, $\{K_1, K_2, K_3\}$ to be the link diagrams which correspond to links $\mathbf{K}$, $\{\mathbf{K}_1, \mathbf{K}_2\}$, $\{\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3\}$ respectively. The link invariants $v_1, v_2, v_{3.1}, v_{3.2}, v_{4.1}, v_{4.2}, v_{4.3}$ and $v_{4.4}$ have the explicit combinatorial expressions as follows:

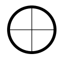
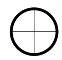
- $v_1(\{\mathbf{K}_1, \mathbf{K}_2\}) = \Big\langle G(\{K_1, K_2\}),$  $\Big\rangle_\chi,$ (3.1)
- $v_2(\mathbf{K}) = -\frac{1}{6} + \Big\langle \bar{G}(K),$  $\Big\rangle_\chi,$ (3.2)
- $v_{3.1}(\mathbf{K}) = \Big\langle G(K),\ 2$  $+$  $+ \frac{1}{2}$  $\Big\rangle_\chi - I_{3.1}(K),$ (3.3)
- $v_{3.2}(\{\mathbf{K}_1, \mathbf{K}_2\}) = \Big\langle G(\{K_1, K_2\}),$  $+$  $+ \frac{1}{3}$  $\Big\rangle_\chi$
  $-I_{3.2}(\{K_1, K_2\})$ (3.4)
- $v_{4.1}(\mathbf{K}) = \Big\langle \bar{G}(K),$  $+$  $+ 2$  $+ 4$  $+ 5$  $+ 7$  $\Big\rangle_\chi$
  $+\Big\langle \bar{G}(K), \frac{1}{6}$  $+ \frac{1}{2}$  $+ 2$  $+ 2$  $\Big\rangle_\chi$
  $-I_{4.1.1}(K) - I_{4.1.2}(K) + \frac{1}{360},$ (3.5)
- $v_{4.2}(\mathbf{K}) = \Big\langle \bar{G}(K),$  $+$  $+$  $+ \frac{1}{2}$  $- \frac{1}{6}$  $\Big\rangle_\chi$
  $-I_{4.2}(K) - \frac{1}{360},$ (3.6)
- $v_{4.3}(\{\mathbf{K}_1, \mathbf{K}_2\})$
  $= \Big\langle \bar{G}(\{K_1, K_2\}),$  $+$  $+ 2$  $+$  $\Big\rangle_\chi$
  $+\Big\langle \bar{G}(\{K_1, K_2\}),$  $+$  $+ \frac{1}{2}$  $+ \frac{1}{2}$  $\Big\rangle_\chi$
  $-I_{4.3.1}(\{K_1, K_2\}) - I_{4.3.2}(\{K_1, K_2\}),$ (3.7)

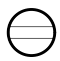
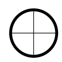
- $v_{4.4}(\{\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3\}) = \Big\langle \bar{G}(\{K_1, K_2, K_3\}),$  $+$  $+$  $\Big\rangle_\chi$
$-I_{4.4}(\{K_1, K_2, K_3\}),$ (3.8)

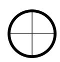
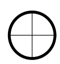
where $\bar{G}(L) = G(L) - G(\alpha(L))$. Set $R = G \circ \alpha \circ S \circ Q$ and $\bar{P}(\{L : a, b\}) = P(\{L : a, b\}) - P(\{\alpha(L) : a, b\})$. Here $I_{3.1}, I_{3.2}, I_{4.1.1}, I_{4.1.2}, I_{4.2}, I_{4.3.1}, I_{4.3.2}, I_{4.4}$ are given as follows:

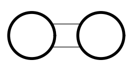
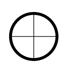
- $I_{3.1}(K) = \sum_a \Big\langle P(\{K : a\}),$  $\Big\rangle_\chi \Big\langle R(\{K : a\}, [\gamma] - [\alpha]),$  $\Big\rangle_\chi,$ (3.9)

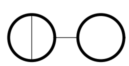
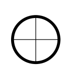
- $I_{3.2}(\{K_1, K_2\}) = \sum_a \Big\langle (P(\{K_1, K_2 : a\}),$  $\Big\rangle_\chi$
$\times \Big\langle R(\{K_1, K_2 : a\}, [\alpha] - [\gamma]),$  $\Big\rangle_\chi,$

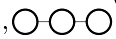
- $I_{4.1.1}(K) = \sum_{(a_1,a_2)} \Big\langle \bar{P}(\{K : a_1, a_2\}),$  $\Big\rangle_\chi$
$\times \Big\langle R(\{K : a_1, a_2\}, 3[\gamma, \gamma] - 2[\alpha, \gamma] - 2[\gamma, \alpha] + [\beta, \beta]),$  $\Big\rangle_\chi,$

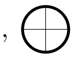
- $I_{4.1.2}(K) = \sum_{(a_1,a_2)} \Big\langle \bar{P}(\{K : a_1, a_2\}),$  $\Big\rangle_\chi$
$\times \Big\langle R(\{K : a_1, a_2\}, [\gamma, \gamma] - [\alpha, \gamma] - [\gamma, \alpha] + [\alpha, \alpha]),$  $\Big\rangle_\chi,$

- $I_{4.2}(K) = \sum_{(a_1,a_2)} \Big\langle \bar{P}(\{K : a_1, a_2\}),$  $\Big\rangle_\chi$
$\times \Big\langle R(\{K : a_1, a_2\}, [\gamma, \gamma] - [\alpha, \gamma] - [\gamma, \alpha] + [\beta, \beta]),$  $\Big\rangle_\chi,$

- $I_{4.3.1}(\{K_1, K_2\}) = \sum_{(a_1,a_2)} \Big\langle \bar{P}(\{K_1, K_2 : a_1, a_2\}),$  $\Big\rangle_\chi$
$\times \Big\langle R(\{K_1, K_2 : a_1, a_2\}, [\gamma, \gamma] - [\beta, \beta]),$  $\Big\rangle_\chi,$

- $I_{4.3.2}(\{K_1, K_2\}) = \sum_{(a_1,a_2)} \Big\langle \bar{P}(\{K_1, K_2 : a_1, a_2\}),$  $\Big\rangle_\chi$
$\times \Big\langle R(\{K_1, K_2 : a_1, a_2\}, [\alpha, \gamma] + [\gamma, \alpha] - [\gamma, \gamma] - [\alpha, \alpha]),$  $\Big\rangle_\chi,$

- $I_{4.4}(\{K_1, K_2, K_3\}) = \sum_{(a_1,a_2)} \Big\langle \bar{P}(\{K_1, K_2, K_3 : a_1, a_2\}),$  $\Big\rangle_\chi$

$\times \Big\langle R(\{K_1, K_2, K_3 : a_1, a_2\}, [\gamma, \gamma] + [\alpha, \alpha] - [\alpha, \gamma] - [\gamma, \alpha]),$  $\Big\rangle_\chi,$

where the sum $\sum_a$ $\Big($resp. $\sum_{(a_1,a_2)}\Big)$ is taken over all the crossings (resp. all the unordered pairs of the crossings). □

Remark. The Gauss diagram formulas in Theorem 2 is expressed by the pairing $\langle \hat{G}, \hat{D} \rangle_\chi$ in Definition 3.5. So it is easy to compute the link invariants $v_1, v_2, v_{3.1}, v_{3.2}, v_{4.1}, v_{4.2}, v_{4.3}, v_{4.4}$ for any link. □

See section 5 for the proof of Theorem 2.

Conjecture 3.13. There exist Gauss diagram formulas for any Vassiliev invariants of any degree. □

4. Homfly Polynomial and Some Caluculations

4.1. *Relation to Homfly Polynomial*

In this section, we shall discuss the relation between Theorem 2 and Homfly polynomial.

Definition 4.1. (Homfly polynomial) For a link diagram L, the Homfly polynomial $P_L(t, z)$ is characterized by the skein relation:

$$tP_{L_+}(t, z) - t^{-1}P_{L_-}(t, z) = zP_{L_0}(t, z)$$

$$P_U \quad = \quad 1, \tag{4.1}$$

where U denotes a trivial knot. The links L_+, L_-, L_0 are given in Figure 8. □

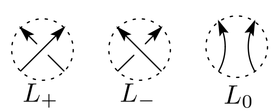


Fig. 8. skein relation

It is known that the Kontsevich integral and the weight system of $su(N)$ gives the Homfly polynomial. More precisely, the following fact holds.

Fact 4.2. Let $\mathbf{L} = \{\mathbf{K}_1, \cdots, \mathbf{K}_n\}$ be a link. Define $\hat{P}_{\mathbf{L}}(x, N)$ by

$$\hat{P}_{\mathbf{L}}(x, N) = N^{n-1} \exp\Big(-x\frac{N^2 - 1}{2N} w(\mathbf{L})\Big) \frac{\hat{Z}_W(\mathbf{L})}{\hat{Z}_W(U)}, \tag{4.2}$$

where $w(\mathbf{L})$ is given by

$$w(\mathbf{L}) = \sum_{1 \le i < j \le n} v_1(\{\mathbf{K}_i, \mathbf{K}_j\}).$$

Let $L = \{K_1, \cdots, K_n\}$ be the link diagram of $\mathbf{L}$. Then,

$$P_L(e^{\frac{Nx}{2}}, e^{\frac{x}{2}} - e^{-\frac{x}{2}}) = \hat{P}_{\mathbf{L}}(x, N)$$

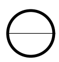
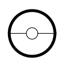
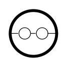
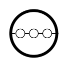
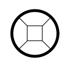
holds. $\Box$

Since $v_1, v_2, v_{3.1}, v_{3.2}, v_{4.1}, v_{4.2}, v_{4.3}, v_{4.4}$ are link invariants and depend only on its link diagrams, we write $v_1(\{K_i, K_j\})$ instead of $v_1(\{\mathbf{K}_i, \mathbf{K}_j\})$, etc. From Theorem 2 and Fact 4.2, we immediately obtain the following corollary.

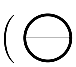
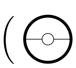
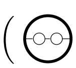
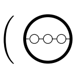
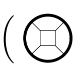
Corollary 1. Up to degree four, the power series expansion of Homfly polynomial with respect to x has the explicit Gauss diagram formula as follows:

$$\begin{aligned}&\Big[P_L(e^{\frac{Nx}{2}}, e^{\frac{x}{2}} - e^{-\frac{x}{2}})\Big]^{(4)} \\ &\quad = W^{(4)}_{su(N)}\left(N^{n-1}\Big\{\exp\Big(\sum_{D \in \bar{\mathfrak{D}}_K} D\ u(D : L)\Big)\Big\}\Big\{\sum_{D \in \mathfrak{D}_L} D\ w(D : L)\Big\}\right), \qquad (4.3)\end{aligned}$$

where $W^{(4)}_{su(N)}(D) = \big[W_{su(N)}(D)\big]^{(4)}$. The first sum $\sum_{D \in \bar{\mathfrak{D}}_K}$ is taken over the following CC diagrams:

$\bar{\mathfrak{D}}_K = \Big\{$  ,  ,  ,  ,  $\Big\}$.

The second sum $\sum_{D \in \mathfrak{D}_L}$ is taken over the same CC diagrams as (2.5). Here $w(D : L)$ is the same as (2.6) and $u(D : L)$ is given as follows:

- $u\Big($  $: L\Big) = -\sum_{1 \le i < j \le n} v_1(\{K_i, K_j\}),$
- $u\Big($  $: L\Big) = (-\frac{1}{2})\{\frac{1}{6} + \sum_{i=1}^{n} v_2(K_i)\},$
- $u\Big($  $: L\Big) = (-\frac{1}{2})^2 \sum_{i=1}^{n} v_{3.1}(K_i),$
- $u\Big($  $: L\Big) = (-\frac{1}{2})^3\{-\frac{1}{360} + \sum_{i=1}^{n} v_{4.1}(K_i)\},$
- $u\Big($  $: L\Big) = \sum_{i=1}^{n}\{\frac{1}{360} + v_{4.2}(K_i)\}.$

$\Box$

4.2. *Some caluculations*

We give an example of Theorem 2 (Gauss diagram formula). As an example, we compute $v_2, v_{3.1}, v_{4.1}, v_{4.2}$ for a knot diagram K given in Figure 9.

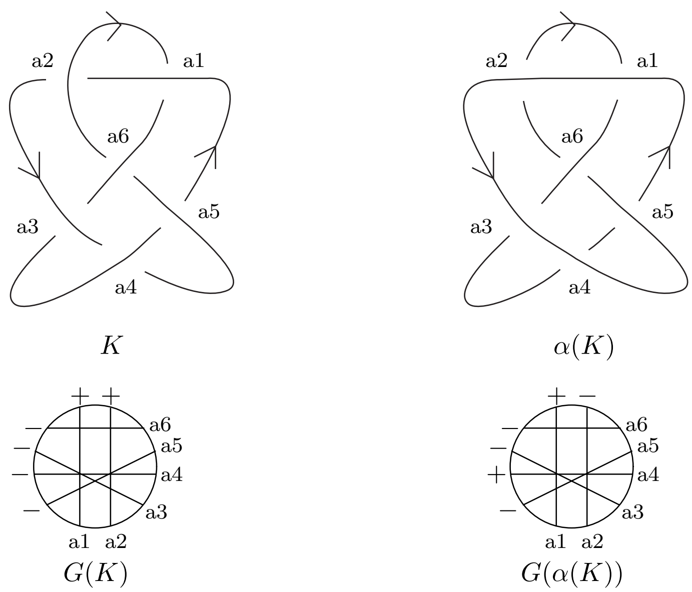


Fig. 9. the knot diagram K, $\alpha(K)$ and their Gauss diagram $G(K)$, $G(\alpha(K))$

4.2.1. $v_2(K)$

Using the Gauss diagram $G(K)$, $G\big(\alpha(K)\big)$ in Figure 9, we get

$$\Big\langle G(K), \bigoplus \Big\rangle_\chi = -5, \qquad \Big\langle G(\alpha(K)), \bigoplus \Big\rangle_\chi = -1. \tag{4.4}$$

Inserting these into Gauss diagram formula (3.2), we obtain

$$\begin{aligned} v_2(K) &= -\frac{1}{6} + \Big\langle \bar{G}(K), \bigoplus \Big\rangle_\chi \\ &= -\frac{1}{6} + \Big\langle G(K), \bigoplus \Big\rangle_\chi - \Big\langle G(\alpha(K)), \bigoplus \Big\rangle_\chi \\ &= -\frac{1}{6} - 4. \end{aligned} \tag{4.5}$$

Remark. Notice that we cannot replace $\alpha(K)$ by a trivial knot U in the second equation of (4.4) since $\Big\langle G(\alpha(K)), \bigoplus \Big\rangle_\chi$ is not a knot invariant.

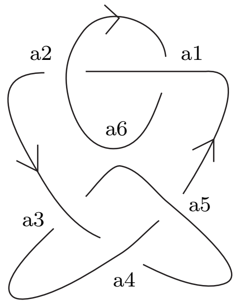
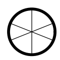
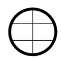
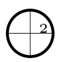


Fig. 10. link diagram $Q(\{K : \mathrm{a}6\}, [\alpha])$

4.2.2. $v_{3.1}(K)$

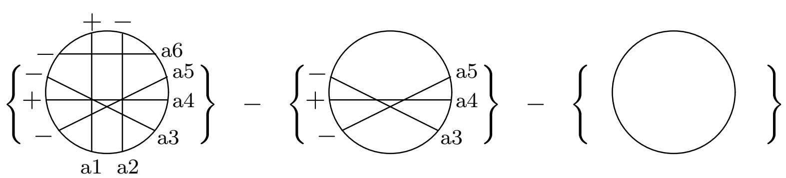
Using the Gauss diagram $G(K)$ in Figure 9, we get

$\Big\langle G(K),$  $\Big\rangle_{\chi} = 5, \quad \Big\langle G(K),$  $\Big\rangle_{\chi} = 2,$

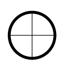
$\Big\langle G(K),$  $\Big\rangle_{\chi} = -6.$ (4.6)

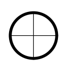
Next we shall caluculate $I_{3.1}(K)$. Considering Figure 10, we have

$R(\{K : \mathrm{a}_6\}, [\gamma] - [\alpha])$

$=$ 

We can caluculate the other $R(\{K : \mathrm{a}_i\}, [\gamma] - [\alpha])$ $(i = 1, \cdots, 5)$ in the same way. Then we have

$\Big\langle R(\{K : \mathrm{a}_i\}, [\gamma] - [\alpha]),$  $\Big\rangle_{\chi} = -1 \quad (i = 1, \cdots, 5),$

$\Big\langle R(\{K : \mathrm{a}_6\}, [\gamma] - [\alpha]),$  $\Big\rangle_{\chi} = 0.$

Inserting these into (3.9), we obtain

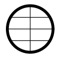
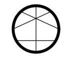
$$I_{3.1}(K) = 1. \tag{4.7}$$

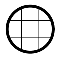
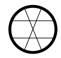
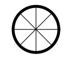
Inserting (4.6) (4.7) into Gauss diagram formula (3.3) yields

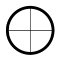
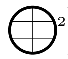
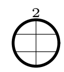
$$v_{3.1}(K) = 8. \tag{4.8}$$

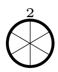
4.2.3. $v_{4.1}(K)$ *and* $v_{4.2}(K)$

Using the Gauss diagram $G(K)$, $G\big(\alpha(K)\big)$ in Figure 9, we get

$\Big\langle \bar{G}(K),$  $\Big\rangle_\chi = 0,$ $\Big\langle \bar{G}(K),$  $\Big\rangle_\chi = 0,$ $\Big\langle \bar{G}(K),$  $\Big\rangle_\chi = -6,$

$\Big\langle \bar{G}(K),$  $\Big\rangle_\chi = 4,$ $\Big\langle \bar{G}(K),$  $\Big\rangle_\chi = 2,$ $\Big\langle \bar{G}(K),$  $\Big\rangle_\chi = -2,$

$\Big\langle \bar{G}(K),$  $\Big\rangle_\chi = -4,$ $\Big\langle \bar{G}(K),$  $\Big\rangle_\chi = -20,$ $\Big\langle \bar{G}(K),$  $\Big\rangle_\chi = 12,$

$\Big\langle \bar{G}(K),$  $\Big\rangle_\chi = 0,$

$I_{4.1.1}(K) = 4,$ $\qquad I_{4.1.2}(K) = -2,$ $\qquad I_{4.2}(K) = -2.$

Inserting these equation into Gauss diagram formula (3.5) and (3.6) yields

$$v_{4.1}(K) = \frac{1}{360} + \frac{34}{3}, \qquad v_{4.2}(K) = -\frac{1}{360} + \frac{38}{3}. \tag{4.9}$$

4.2.4.

Inserting (4.5),(4.8),(4.9) into the right side of (4.3), we get

$$\Big\{\text{the right side of (4.3)}\Big\}$$
$$= 1 + (N^2-1)x^2 + N(N^2-1)x^3 + \frac{-13+6N^2+7N^4}{12}x^4. \tag{4.10}$$

The Homfly polynomial of K is caluculated by the skein relation (4.1):

$$P_K(t,z) = t^4z^2 + t^4 - t^2z^4 - 3t^2z^2 - 2t^2 + z^2 + 2.$$

We can easiliy check that $\big[P_K(e^{\frac{Nx}{2}}, e^{\frac{x}{2}} - e^{-\frac{x}{2}})\big]^{(4)}$ coincides with (4.10).

5. Proof of Theorem 2

In this section we shall derive the Gauss diagram formula from Kontsevich integral (Proof of Theorem 2).

5.1. *Sketch of the Proof of Theorem 2*

We begin by briefly sketching the proof of Theorem 2.

The integrand of $\langle\!\langle \mathbf{L}, D\rangle\!\rangle$ in $v_1, v_2, v_{3.1}, v_{3.2}, v_{4.1}, v_{4.2}, v_{4.3}, v_{4.4}$ (see Definition 2.8 and (2.7) $\sim$ (2.11)) has the following form:

$$d\log(z_{i_kj_k}(t_k)) = di\theta_{i_kj_k}(t_k) + d\log r_{i_kj_k}(t_k), \tag{5.1}$$

where $\theta_{i_kj_k}(t_k)$ and $r_{i_kj_k}(t_k)$ are defined by the polar form

$$z_{i_kj_k}(t_k) = r_{i_kj_k}(t_k)\exp(i\theta_{i_kj_k}(t_k)).$$

We expand the integrand of $\langle\langle \mathbf{L}, D\rangle\rangle$ according to (5.1). For example, if the degree of D is two, the integrand of $\langle\langle \mathbf{L}, D\rangle\rangle$ is expanded as follows:

$$\begin{aligned}\prod_{k=1}^{2}\Big\{\epsilon\ d\log(z_{i_kj_k}(t_k))\Big\} &= \Big\{\epsilon\ di\theta_{i_1j_1}(t_1)\Big\}\Big\{\epsilon\ di\theta_{i_2j_2}(t_2)\Big\} \\ &\quad+\Big\{\epsilon\ di\theta_{i_1j_1}(t_1)\Big\}\Big\{\epsilon\ d\log r_{i_2j_2}(t_2)\Big\} \\ &\quad+\Big\{\epsilon\ d\log r_{i_1j_1}(t_1)\Big\}\Big\{\epsilon\ di\theta_{i_2j_2}(t_2)\Big\} \\ &\quad+\Big\{\epsilon\ d\log r_{i_1j_1}(t_1)\Big\}\Big\{\epsilon\ d\log r_{i_2j_2}(t_2)\Big\}.\end{aligned} \tag{5.2}$$

Without loss of generality, we may replace the link $\mathbf{L}$ with the link $A^b(L)$ in a nice position to caluculate (see Definition 5.4). Then key observation is as follows.

- The integrals which have odd number of $d\log r_{i_kj_k}(t_k)$'s are pure imaginary and do not contribute to the caluculation, since the Kontsevich integral is real valued (See Lemma 5.3). For example, the second and third terms in the right side of (5.2) do not contribute to the caluculation.
- The part of $di\theta_{i_kj_k}(t_k)$ integral is localized around the cylinders (crossings) of the link $A^b(L)$, since $\theta_{i_kj_k}(t_k)$ does not vary on the plane $\mathbb{R}\times\{0\}\times\mathbb{R} = \{(x_1,0,x_3)\in\mathbb{R}^3\}$. Thus it is easy to evaluate (See Lemma 5.10 and Lemma 5.12). For example, the first term in the right hand side of (5.2) is easily caluculated.
- The part of $d\log r_{i_kj_k}(t_k)$ integral is difficult to evaluate. But we can avoid this $d\log r_{i_kj_k}(t_k)$ integral as follows. First, since $r_{i_kj_k}(t_k)$ takes the same value for both signature $\pm$ of the cylinder, the part of $d\log r_{i_kj_k}(t_k)$ integral does not depend on the signatures of the link $A^b(L)$. In other words, it essentially depends only on its projection to $\mathbb{R}\times\{0\}\times\mathbb{R} = \{(x_1,0,x_3)\in\mathbb{R}^3\}$ and takes the same value for L and $\alpha(L)$. For example, the fourth term in the right hand side of (5.2) does not depend on the signatures of the link $A^b(L)$. Second, the modified Kontsevich integral is a link invariant. These two point leads to the final Gauss diagram formula.

5.2. *Preparation for the proof of Theorem 2*

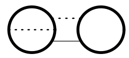
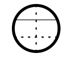
In this section, we shall fix notations for the proof of Theorem 2.

5.2.1.

Definition 5.1. (Dotted Diagram) An IL diagram $\{D,\kappa\}$ is called a *Dotted diagram* if $\kappa(c)=0,1$ $(c\in C(D))$. A chord c is called a *normal chord* if $\kappa(c)=1$ and a *dotted chord* if $\kappa(c)=0$. In figures, we draw a normal chord ($\kappa(c)=1$) by a thin line and a dotted chord ($\kappa(c)=0$) by a dotted line as follows:

———— normal chord ($\kappa(c)=1$), ·············· dotted chord ($\kappa(c)=0$).

We give two exmples of dotted diagrams,

, .

Definition 5.2. For a complex number z, define $\theta(z)$ and $r(z)$ by the polar form $z = r\exp(i\theta)$. Let $\mathbf{L}$ be a link and $\hat{D}$ a dotted diagram of degree m which has l-normal chords and $(m-l)$-dotted chords, where l is fixed. We consider m-planes $t = t_k, (k = 1, \cdots, l, \cdots, m)$ where $t_{\min} < t_1 < \cdots < t_l < t_{\max}$ and $t_{\min} < t_{l+1} < \cdots < t_m < t_{\max}$. For $1 \leq k \leq m$, set $(\pi^{-1})(t_k) = \{s_k^1, \cdots, s_k^{n(t_k)}\}$, where $n(t_k)$ denotes the number of points on the section $t = t_k$ of the link $\mathbf{L}$. For $1 \leq k \leq l$, set $\theta_{ij}(t_k) = \theta \circ z\big\{\vec{x}(s_k^i) - \vec{x}(s_k^j)\big\}$. For $(l+1) \leq k \leq m$, set $r_{ij}(t_k) = r \circ z\big\{\vec{x}(s_k^i) - \vec{x}(s_k^j)\big\}$. Define the collection of all pairings by $P = \big\{(i_1, j_1), \cdots, (i_m, j_m) : 1 \leq i_k \leq j_k \leq n(t_k) \quad (k = 1, \cdots, m)\big\}$. For $p \in P$, we shall define a dotted diagram D_p of degree m. For all k, we join $s_k^{i_k}$ and $s_k^{j_k}$ by normal chords if $1 \leq k \leq l$, and join $s_k^{i_k}$ and $s_k^{j_k}$ by dotted chords if $(l+1) \leq k \leq m$ on X. Define D_p to be the resulting dotted diagram of degree m which has l-normal chords and $(m-l)$-dotted chords. Define $\Big\langle \mathbf{L}, \hat{D} \Big\rangle$ by

$$\Big\langle \mathbf{L}, \hat{D} \Big\rangle = \frac{1}{(i\pi)^m} \int_{\substack{t_{\max} > t_1 > \cdots > t_l > t_{\min} \\ t_{\max} > t_{l+1} > \cdots > t_m > t_{\min}}} \sum_{p \in P} \prod_{k=1}^{l} \{\epsilon \; di\theta_{i_k j_k}(t_k)\} \times \prod_{k=l+1}^{m} \{\epsilon \; d\log r_{i_k j_k}(t_k)\} \Theta(D_P, \hat{D}), \quad (5.3)$$

where the sum is taken over all the pairings $p \in P$. $\Theta(D_p, \hat{D})$ is defined by

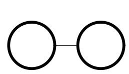
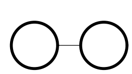
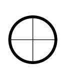
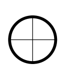
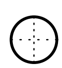
$$\Theta(D_p, \hat{D}) = \begin{cases} 1 & \text{if} \;\; D_p = \hat{D} \\ 0 & \text{if} \;\; D_p \neq \hat{D} \end{cases}.$$

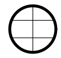
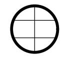
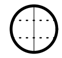
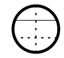
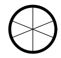
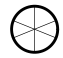
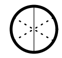
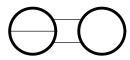
Let $\hat{D}_i$ be a dotted chord diagram. More generally, for a formal linear combination of dotted diagrams $\sum_i b_i \; \hat{D}_i$ $(b_i \in \mathbf{C})$, set

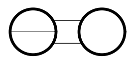
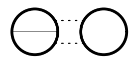
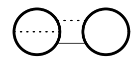
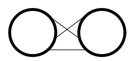
$$\Big\langle \mathbf{L}, \sum_i b_i \; \hat{D}_i \Big\rangle = \sum_i b_i \; \Big\langle \mathbf{L}, \hat{D}_i \Big\rangle.$$

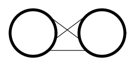
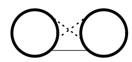
Remark. Roughly speaking, a normal chord represents "$di\theta$" integral, and a dotted chord represents "$d\log r$" integral.

Lemma 5.3.

- Re $\Big\langle\!\Big\langle \{\mathbf{K}_1, \mathbf{K}_2\},$  $\Big\rangle\!\Big\rangle = \Big\langle \{\mathbf{K}_1, \mathbf{K}_2\},$  $\Big\rangle$, (5.4)
- Re $\Big\langle\!\Big\langle \mathbf{K},$  $\Big\rangle\!\Big\rangle = \Big\langle \mathbf{K},$  $+$  $\Big\rangle$, (5.5)

- $\mathrm{Re}\,\langle\!\langle \mathbf{K},$  $\rangle\!\rangle = \langle \mathbf{K},$  $+$  $+$  $\rangle,$ (5.6)
- $\mathrm{Re}\,\langle\!\langle \mathbf{K},$  $\rangle\!\rangle = \langle \mathbf{K},$  $+$  $\rangle,$ (5.7)
- $\mathrm{Re}\,\langle\!\langle \{\mathbf{K}_1, \mathbf{K}_2\},$  $\rangle\!\rangle$

  $= \langle \{\mathbf{K}_1, \mathbf{K}_2\},$  $+$  $+$  $\rangle,$ (5.8)
- $\mathrm{Re}\,\langle\!\langle \{\mathbf{K}_1, \mathbf{K}_2\},$  $\rangle\!\rangle$

  $= \langle \{\mathbf{K}_1, \mathbf{K}_2\},$  $+$  $\rangle,$ (5.9)

where $\mathrm{Re}\,\langle\!\langle K, D\rangle\!\rangle$ denotes the real part of complex number $\langle\!\langle K, D\rangle\!\rangle$.

Proof. We expand the integrand of $\langle\!\langle K, D\rangle\!\rangle$ according to (5.1). Only the integrals which have even number of $d\log r_{i_k j_k}(t_k)$'s contribute, since the integrals which have odd number of $d\log r_{i_k j_k}(t_k)$'s are pure imaginary. This proves the above lemma. $\square$

5.2.2.

Definition 5.4. (AF Link) Let L be a link diagram in $\mathbb{R}\times\{0\}\times\mathbb{R} = \{(x_1, 0, x_3) \in \mathbb{R}^3\}$. Without loss of generality, we may assume that the two curves around each crossing are given by

$$\left\{ \begin{array}{c} x_3 = \frac{\pi}{2}(-x_1 + b) \\ x_2 = 0 \end{array} \right. \quad (-b \le x_1 \le b), \quad \left\{ \begin{array}{c} x_3 = \frac{\pi}{2}(x_1 + b) \\ x_2 = 0 \end{array} \right. \quad (-b \le x_1 \le b) \tag{5.10}$$

with some parallel transformation (see the left side of Figure 11). Here b is sufficiently small. In other words, this assumption is that two curves around the crossing are on the diagonal lines of some sufficiantly small rectangle parallel to t-axis.

For this link diagram L, we shall define a link $A^b(L)$ called the *Almost-Flat Link* (AF Link) of L as follows. For each crossing of L, we replace the two curves (5.10) by

$$\left\{ \begin{array}{c} x_1 = b\cos(x_3/b) \\ x_2 = b\sin(x_3/b) \end{array} \right. \quad (0 \le x_3 \le b\pi), \quad \left\{ \begin{array}{c} x_1 = -b\cos(x_3/b) \\ x_2 = -b\sin(x_3/b) \end{array} \right. \quad (0 \le x_3 \le b\pi), \tag{5.11}$$

or

$$\left\{ \begin{array}{c} x_1 = b\cos(x_3/b) \\ x_2 = -b\sin(x_3/b) \end{array} \right. \quad (0 \le x_3 \le b\pi), \quad \left\{ \begin{array}{c} x_1 = -b\cos(x_3/b) \\ x_2 = b\sin(x_3/b) \end{array} \right. \quad (0 \le x_3 \le b\pi) \tag{5.12}$$

according to the signature of the crossing (see Figure 11). In other words, we replace the two curves on the rectangle by two curves winding around the cylinder so that projecting two curves winding around the cylinder to $\mathbb{R}\times\{0\}\times\mathbb{R}$ yields the

signature of the crossing . For sufficiently small b, we define the AF link $A^b(L)$ to be the resulting link.

Since the cylinders of the AF link $A^b(L)$ is one-to-one correspondent to the crossings of the link diagram L, we define the signature of each cylinder to be the signature of the corresponding crossing.

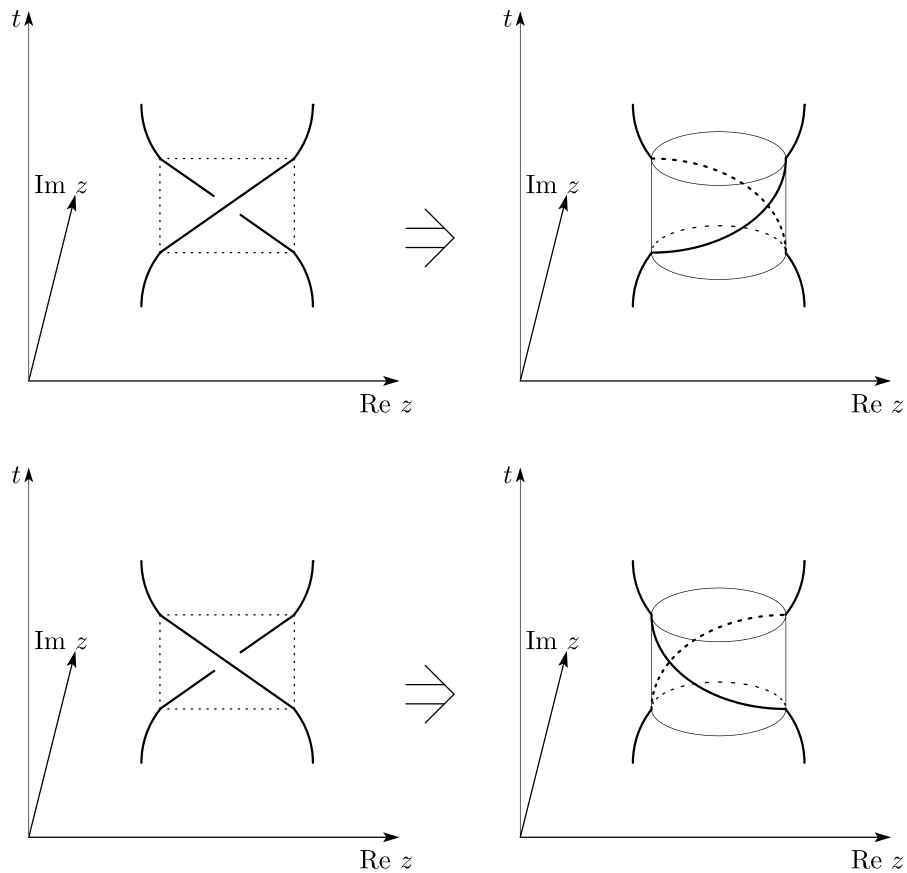
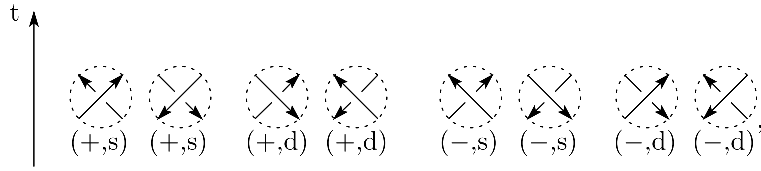


Fig. 11. L and $A^b(L)$

Definition 5.5. (Direction of the Crossing) Let L be a link diagram in $\mathbb{R} \times \{0\} \times \mathbb{R} = \{(x_1, 0, x_3) \in \mathbb{R}^3\}$ as explained in Definition 5.4. We assign a signature $\pm$ to each crossing of L as usual (Definiton 3.1). Moreover, we assign "s" ("s" denotes "same") to each crossing of L if the directions of the two arrows (orientations) are the same with respect to t-axis, and assign "d" if they are different ("d" denotes

"different"):



where "s", "d" are formal letters. We call "s","d" the *Direction of the Crossing*.

Remark. Of courese, the concept of direction of the crossing depend on how to choose t-axis. So, it is not the proper quantity of link diagrams. Although the concept of directions appears in the computaion, it disappears in the final result (see Theorem 2). □

We shall extend the definition of IL diagram, Gauss diagram, ML diagram and the pairing $\langle \hat{G}, \hat{D} \rangle_\chi$ to include the concept of direciton (see Definition 3.2, Definition 3.3, Definition 3.4, Definition 3.5).

Definition 5.6. (*IDL Diagram*) Let D be a chord diagram, and let $C(D)$ be the set of all chords of D. By a direction-labelling of D, we mean a map $f : C(D) \to \{s, d, n\}$, where s, d, n are formal letters. *An Integer-Direction-Labeled Chord Diagram* (IDL Diagram) is a triple $\{D, \kappa, f\}$ of a chord diagram D together with an integer-labelling κ and a direction-labelling f. Two IDL diagrams $\{D, \kappa, f\}$, $\{D', \kappa', f'\}$ are regarded as equal if D, D' are equal as chord diagrams and the homeomorphism $F : D \to D'$ preserves integer-labelling $\kappa'(F(c)) = \kappa(c)$ $(c \in C(D))$ and direction-labelling $f'(F(c)) = f(c)$ $(c \in C(D))$.

Definition 5.7. (Extended Gauss Diagram) An IDL diagram $\{G, \epsilon, f\}$ is called a *Extended Gauss Diagram* if $\epsilon(c) = \pm 1$ and $f(c) = s, d$ (c $\in C(D)$).

Let L be a link diagram as explained in Definition 5.4. and let $\{L : a_1, \cdots, a_m\}$ be a link diagram L where we select some distinct crossings $a_1, \cdots, a_m$ out of all crossings of L. Define a extended Gauss diagram $P^e(\{L : a_1, \cdots, a_m\})$ as follows. For each a_i, set $\vec{y}^{-1}(a_i) = \{s(a_i), s'(a_i)\}$ as the inverse image of a_i. For each crossing a_i, we join $s(a_i), s'(a_i)$ by a chord on X and label this chord by the signature and the direction of a_i $(i = 1, \cdots, m)$. We define an extended Gauss diagram $P^e(\{L : a_1, \cdots, a_m\})$ to be the result.

Specially, If $\{a_1, \cdots, a_m\}$ are all the crossings of L (this means we select all the crossings of L) , we write $G^e(L) = P^e(\{L : a_1, \cdots, a_m\})$ and call it the extended Gauss diagram of L. For example, see Fig 12.

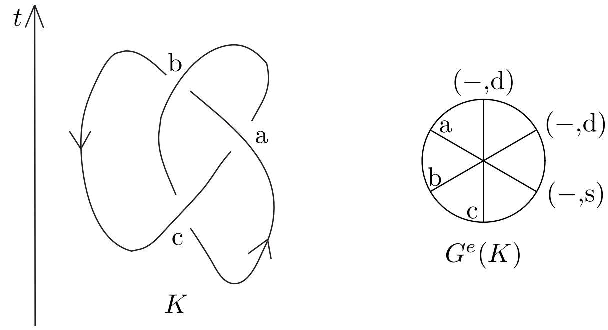


Fig. 12.

Definition 5.8. (Extended ML Diagram) An IDL diagram $\{D, m, \Delta\}$ is called a *Extended Multiplicity-Labeled Diagram* (Extended ML Diagram) if $m(c) = 1, 2$ and $\Delta(c) = s, d, n$ $(c \in C(D))$. In figures, we draw a chord as follows.

______	$m(c) = 1, \Delta(c) = n,$	___2___	$m(c) = 2, \Delta(c) = n$
___s___	$m(c) = 1, \Delta(c) = s,$	___2s___	$m(c) = 2, \Delta(c) = s$
___d___	$m(c) = 1, \Delta(c) = d,$	___2d___	$m(c) = 2, \Delta(c) = d$

We give two examples of ML diagrams.

Definition 5.9. Let $\hat{G}^e = \{G, \epsilon, f\}$ be an extended Gauss diagram and $\hat{D}^e = \{D, m, \Delta\}$ be a extended ML diagram. Let $\psi : D \to G$ be an embedding of D into G which maps the circles of D to those of G preserving the orientations and maps each chord of D to a chord of G. Let $C(G)$ be the set of all the chords of G. For ψ, define a map $\kappa^e_\psi : C(G) \to \{0, 1, 2\} \times \{s, d, n\}$ by

$$\kappa^e_\psi(c) = \begin{cases} (m(\psi^{-1}(c)), \Delta(\psi^{-1}(c))) & \text{if } c \in \psi(D) \\ \quad (\;0 \;\; , \;\; n \;) & \text{if } c \notin \psi(D) \end{cases}$$

Two embedding ψ, φ are said to be equal if $\kappa^e_\psi = \kappa^e_\varphi$. The equivalence class of an embedding ψ is denoted by $[\psi]$. Define $\delta : \{s, d, n\} \times \{s, d\} \to \pm 1$ by

$$\delta(s : f) = \begin{cases} 1 & (f = s) \\ 0 & (f = d) \end{cases}, \quad \delta(d : f) = \begin{cases} 0 & (f = s) \\ 1 & (f = d) \end{cases}, \quad \delta(n : f) = \begin{cases} 1 & (f = s) \\ 1 & (f = d) \end{cases} .$$

Let $C(D)$ be the set of all chords of D. Define $\mathcal{E}^e([\psi])$ by

$$\mathcal{E}^e([\psi]) = \prod_{c \in C(D)} \big\{\epsilon(\psi(c))\big\}^{m(c)} \delta(\Delta(c), f(\psi(c))),$$

where the product is taken over all chords of D. Notice this definition is well defined.

Define $\langle \hat{G}^e, \hat{D}^e \rangle_{\chi^e}$ by

$$\langle \hat{G}^e, \hat{D}^e \rangle_{\chi^e} = \sum_{[\psi]} \mathcal{E}^e([\psi]),$$

where the sum is taken over all the distinct equivalence classes $[\psi]$.

Let $\hat{G}^e_i$ be a extended Gauss diagram and $\hat{D}^e_i$ a extended ML diagram. More generally, for formal linear combinations $\sum_i b_i\ \hat{G}^e_i$ and $\sum_j c_j\ \hat{D}^e_j$ $(b_i, c_j \in \mathbf{C})$, set

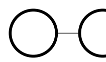
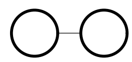
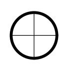
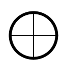
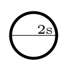
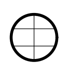
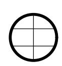
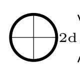
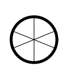
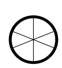
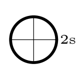
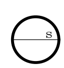
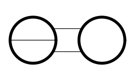
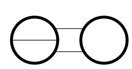
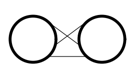
$$\Big\langle \sum_i b_i\ \hat{G}^e_i, \sum_j c_j\ \hat{D}^e_j \Big\rangle_{\chi^e} = \sum_i \sum_j b_i\ c_j\ \langle \hat{G}^e_i, \hat{D}^e_j \rangle_{\chi^e}. \tag{5.13}$$

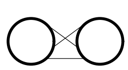
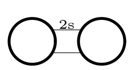
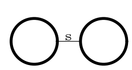
Remark. This definition $\langle \hat{G}^e, \hat{D}^e \rangle_{\chi^e}$ is a natural extension of $\langle \hat{G}, \hat{D} \rangle_\chi$ in Definition 3.5. More precisely, if $\Delta(c) = n$ for all $c \in C(D)$, then

$$\langle \hat{G}^e, \hat{D}^e \rangle_{\chi^e} = \langle \hat{G}, \hat{D} \rangle_\chi,$$

where $\hat{G} = \{G, \epsilon\}$, $\hat{D} = \{D, m\}$.

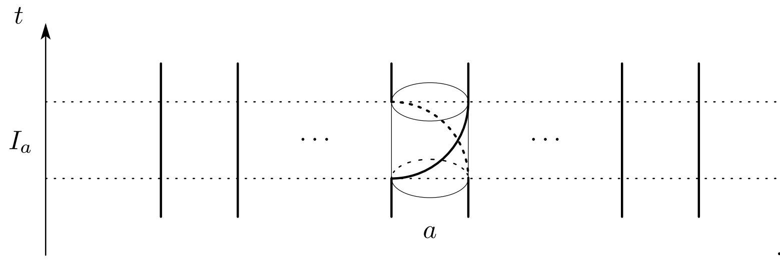
Lemma 5.10. Let $A^b(K)$ and $A^b(\{K_1, K_2\})$ be the AF links which correspond to link diagrams K and $\{K_1, K_2\}$ respectively. Then, for sufficiently small b,

- $\Big\langle A^b(\{K_1, K_2\}),$  $\Big\rangle = \Big\langle G(\{K_1, K_2\}),$  $\Big\rangle_\chi,$ (5.14)
- $\Big\langle A^b(K),$  $\Big\rangle = \Big\langle G^e(K),$  $+ \frac{1}{2}$  $\Big\rangle_{\chi^e} + O(b),$ (5.15)
- $\Big\langle A^b(K),$  $\Big\rangle = \Big\langle G^e(K),$  $+ \frac{1}{2}$  $\Big\rangle_{\chi^e} + O(b),$ (5.16)
- $\Big\langle A^b(K),$  $\Big\rangle = \Big\langle G^e(K),$  $+ \frac{1}{2}$  $+ \frac{1}{3!}$  $\Big\rangle_{\chi^e} + O(b),$ (5.17)
- $\Big\langle A^b(\{K_1, K_2\}),$  $\Big\rangle = \Big\langle G(\{K_1, K_2\}),$  $\Big\rangle_\chi + O(b),$ (5.18)
- $\Big\langle A^b(\{K_1, K_2\}),$  $\Big\rangle$

  $= \Big\langle G^e(\{K_1, K_2\}),$  $+ \frac{1}{2}$  $+ \frac{1}{3!}$  $\Big\rangle_{\chi^e} + O(b).$

  (5.19)

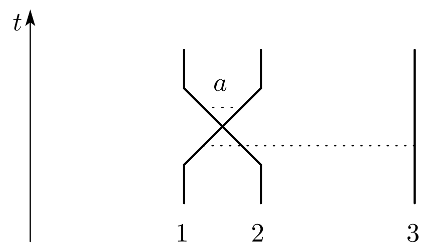
Proof. We shall prove (5.15). We can prove all the other in the same way. In (5.3), $di\theta_{i_k j_k}(t_k)$ integral is localized around the cylinders of the AF knot $A^b(K)$, since $\theta_{i_k j_k}(t_k)$ does not vary on the plane $\mathbb{R} \times \{0\} \times \mathbb{R} = \{(x_1, 0, x_3) \in \mathbb{R}^3\}$. Let a be a cylinder of $A^b(K)$ and I_a the small interval on t-axis which contains the cylinder a. We assume I_a contains only one cylinder a and the other curves in $t \in I_a$ is straight lines parallel to t-axis as follows:



In the above figure, we draw the curves of $A^b(K)$ by thick lines. Then, we have

$$\begin{aligned}
&\Big\langle A^b(K),\ \bigoplus\ \Big\rangle \\
&\quad = \sum_{(a,b)} \frac{1}{(i\pi)^2} \int_{t_1 \in I_a, t_2 \in I_b} \sum_{p \in P} \prod_{k=1}^{2} \{\epsilon\ di\theta_{i_k j_k}(t_k)\} \Theta(D_P,\ \bigoplus\ ) \\
&\qquad + \sum_{a} \frac{1}{(i\pi)^2} \int_{t_1 > t_2 \in I_a} \sum_{p \in P} \prod_{k=1}^{2} \{\epsilon\ di\theta_{i_k j_k}(t_k)\} \Theta(D_P,\ \bigoplus\ ),
\end{aligned} \tag{5.20}$$

where the first sum is taken over all the unordered pair of cylinders (a, b) and the second sum is taken over all cylinders a. By *a pairing on cylinder*, we mean a pairing $p \in P$ for which both $\vec{x}(s_k^{i_k})$, $\vec{x}(s_k^{j_k})$ are on the cylinder $(k = 1, 2)$. Only the pairings on cylinder contribute to the caluculatin for the following reason. For example, we consider the second term in the right side of (5.20) and the pairing $\{(i_1, j_1)(i_2, j_2)\} = \{(1, 2)(1, 3)\}$. Assume $\vec{x}(s_1^1)$, $\vec{x}(s_1^2)$, $\vec{x}(s_2^1)$ are on the cylinder and $\vec{x}(s_2^3)$ is on sraight line :



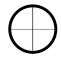
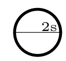
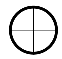
where we draw the curves of $A^b(K)$ by thick lines. Then

$$\begin{aligned}
&\left| \int_{t_1 > t_2 \in I_a} \{\epsilon\ di\theta_{12}(t_1)\}\{\epsilon\ di\theta_{13}(t_2)\} \Theta(D_P,\ \bigoplus\ ) \right| \\
< \ & \left\{ \int_{t_1 \in I_a} \left| d\theta_{12}(t_1) \right| \right\} \left\{ \int_{t_2 \in I_a} \left| d\theta_{13}(t_2) \right| \right\} \\
< \ & (\text{constant}) \times b.
\end{aligned}$$

So the integral correspond to the pairing which is *not* on cylinder is bounded by b.

Anyway since only the pairings on cylinder contribute to the caluculation,

$$\frac{1}{(i\pi)}\int_{t_k\in I_a}\{\epsilon\ di\theta_{i_kj_k}(t_k)\}$$

gives the signature of the cylinder a. Therefore we see that the first term in the right side of (5.20) gives $\Big\langle G(K),$  $\Big\rangle_{\chi}$ and the second term gives $\frac{1}{2}\Big\langle G^e(K),$  $\Big\rangle_{\chi^e}$, considering the restriction of $\Theta(D_P,$  $)$. $\square$

5.2.3.

Definition 5.11. Let $\{A^b(L) : a_1,\cdots,a_l\}$ be a AF link $A^b(L)$ where we select l-cylinders $a_1,\cdots,a_l$ out of all cylinders of $A^b(L)$. Let $\hat{D}$ be a dotted chord diagram of degree $(m+l)$ which has l-normal chords and m-dotted chords. We consider m-planes $t=t_k,(k=1,\cdots,m)$ where $t_{\max}>t_1>\cdots>t_m>t_{\min}$. For $1\le k\le m$, set $(\pi^{-1})(t_k)=\{s_k^1,\cdots,s_k^{n(t_k)}\}$, where $n(t_k)$ denotes the number of points on the section $t=t_k$ of $A^b(L)$. For $1\le k\le m$, set $r_{ij}(t_k)=r\circ z\big\{\vec{x}(s_k^i)-\vec{x}(s_k^j)\big\}$. Define the collection of all pairings by $P=\{(i_1,j_1),\cdots,(i_m,j_m):1\le i_k\le j_k\le n(t_k)\quad(k=1,\cdots,m)\}$.

For each pairing $p\in P$ and the specific cylinders $a_1,\cdots,a_l$, we shall define a dotted diagram $D_{p,a_1\cdots a_l}$ of degree $(m+l)$. For each cylinder a_k $(1\le k\le l)$, we mark two distinct points d_k,d'_k with the same heights on each curves winding around a_k (see Figure 13). We set this height to be $x_3=\frac{b\pi}{2}$ in (5.11) (5.12). Next set $\hat{s}_k=\vec{x}^{-1}(d_k),\hat{s}'_k=\vec{x}^{-1}(d'_k)$ as the inverse images of d_k,d'_k. For $p\in P$ and $a_1,\cdots,a_l$, join $s_k^{i_k}$ and $s_k^{j_k}$ by dotted chords $(k=1,\cdots,m)$ and join $\hat{s}_k$ and $\hat{s}'_k$ by a normal chord $(k=1,\cdots,l)$ on X. Let $D_{p,a_1\cdots a_l}$ be the resulting dotted diagram of degree $(m+l)$ which has l-normal chords and m-dotted chords.

For sufficiently small b, define $\Big[\{A^b(L):a_1,\cdots,a_{m-l}\},\hat{D}\Big]$ by

$$\begin{aligned}&\Big[\{A^b(L):a_1,\cdots,a_l\},\hat{D}\Big]\\&\qquad=\frac{1}{(i\pi)^m}\int_{t_1>\cdots>t_m}\sum_{p\in P}\prod_{k=1}^{m}\{\epsilon\ d\log r_{i_kj_k}(t_k)\}\ \Theta(D_{p,a_1\cdots a_l},\hat{D}),\end{aligned}$$

where the sum is taken over all pairings $p\in P$.

Remark. Roughly speaking, a normal chord represents the cylinder of the AF link, and a dotted chord represents the "$d\log r$" integral.

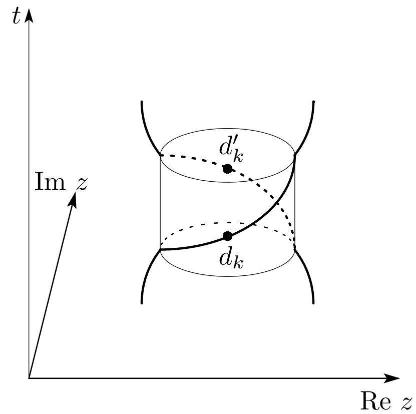


Fig. 13.

Lemma 5.12. Let $A^b(K)$ and $A^b(\{K_1, K_2\})$ be the AF links which correspond to link diagrams K and $\{K_1, K_2\}$ respectively. Then, for sufficiently small b,

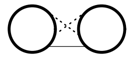
- $$\Big\langle A^b(K), \ \Big\rangle = \sum_a \Big\langle P\big(\{K : a\}\big), \ \Big\rangle_\chi \Big[\{A^b(K) : a\}, \ \Big] + O(b), \tag{5.21}$$

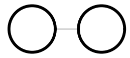
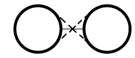
- $$\Big\langle A^b(K), \ \Big\rangle = \sum_a \Big\langle P\big(\{K : a\}\big), \ \Big\rangle_\chi \Big[\{A^b(K) : a\}, \ \Big] + O(b), \tag{5.22}$$

- $$\Big\langle A^b(K), \ \Big\rangle = \sum_a \Big\langle P\big(\{K : a\}\big), \ \Big\rangle_\chi \Big[\{A^b(K) : a\}, \ \Big] + O(b), \tag{5.23}$$

- $$\Big\langle A^b(\{K_1, K_2\}), \ \Big\rangle = \sum_a \Big\langle P\big(\{K_1, K_2 : a\}\big), \ \Big\rangle_\chi \Big[\big\{A^b(\{K_1, K_2\}) : a\big\}, \ \Big] + O(b), \tag{5.24}$$

- $$\Big\langle A^b(\{K_1, K_2\}), \ \Big\rangle = \sum_a \Big\langle P\big(\{K_1, K_2 : a\}\big), \ \Big\rangle_\chi \Big[\big\{A^b(\{K_1, K_2\}) : a\big\}, \ \Big] + O(b), \tag{5.25}$$

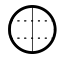
- $\Big\langle A^b(\{K_1, K_2\}),$  $\Big\rangle$

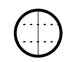
$$= \sum_a \Big\langle P\big(\{K_1, K_2 : a\}\big),$$  $$\Big\rangle_\chi \Big[\{A^b(\{K_1, K_2\}) : a\},$$  $$\Big]$$

$$+O(b), \qquad (5.26)$$

where the sum is taken over all the cylinders a of the AF link. Notice the terms $P\big(\{K : a\}\big)$, etc make sence, since a cylinder of a AF link is identified with the crossing of the corresponding link diagram.

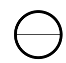
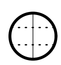
Proof. We shall prove (5.21). We can prove the other in the same way. In (5.3), $di\theta_{i_k j_k}(t_k)$ integral is localized around the cylinders of the AF link $A^b(K)$. So we make the same assumption for the small interval I_a as in the proof of Lemma 5.10. Considering this position, (5.21) becomes:

$$\Big\langle A^b(K),$$  $$\Big\rangle = \sum_a \frac{1}{(i\pi)^3} \int_{t_1 \in I_a} \int_{t_{\max} > t_2 > t_3 > t_{\min}} \sum_{p \in P} \{\epsilon \ di\theta_{i_1 j_1}(t_1)\}$$

$$\times \prod_{k=2}^{3} \{\epsilon \ d \log r_{i_k j_k}(t_k)\} \Theta(D_P,$$  $$).$$

The first sum is taken over all cylinder a. Let P_c be a set of all the pairings $p \in P$ where both $\vec{x}(s_1^{i_1})$ and $\vec{x}(s_1^{j_1})$ are on the cylinder. Only the pairings $p \in P_c$ contribute to the caluculation for the same reason as in the proof of Lemma 5.10. So

$$\frac{1}{(i\pi)} \int_{t_1 \in I_a} \{\epsilon \ di\theta_{i_1 j_1}(t_1)\}$$

gives the signature of the cylinder a, which is equal to $\Big\langle P\big(\{K : a\}\big),$  $\Big\rangle_\chi$. The remaining part gives $\Big[\{A^b(K) : a\},$  $\Big]$. $\square$

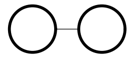
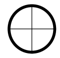
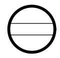
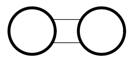
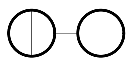
5.2.4.

Definition 5.13. Let $\{L : a_1, \cdots, a_m\}$ be a link diagram L where we select some distinct crossings $a_1, \cdots, a_m$ out of all crossings of L. Define a chord diagram $P_0(\{L : a_1, \cdots, a_m\})$ as follows. For each a_i, set $\vec{y}^{-1}(a_i) = \{s(a_i), s'(a_i)\}$ as the inverse image of a_i. For each crossing a_i, we join $s(a_i), s'(a_i)$ by a chord on X. We define a chord diagram $P_0(\{L : a_1, \cdots, a_m\})$ to be the result. $\square$

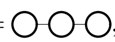
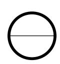
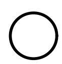
Remark. $P_0(\{L : a_1, \cdots, a_m\})$ is obtained from $P(\{L : a_1, \cdots, a_m\})$ by dropping the signature labelling.

Definition 5.14. We shall define knot diagram $K_\pm^{[1]}, K^{[2]}, K_\pm^{[3]}, \cdots$, etc as follows (see also Figure 14).

- Set $Q(\{K : a\}, [\alpha]) = \{K_+^{[1]}, K_-^{[1]}\}$.

- If $P_0(\{K_1, K_2 : a\}) =$ , set $Q(\{K_1, K_2 : a\}, [\alpha]) = \{K^{[2]}\}$.
- If $P_0(\{K : a_1, a_2\}) =$ , set $Q(\{K : a_1, a_2\}, [\alpha, \gamma]) = \{K_+^{[3]}, K_-^{[3]}\}$,
  $Q(\{K : a_1, a_2\}, [\gamma, \alpha]) = \{K_+^{[4]}, K_-^{[4]}\}$,
  and $Q(\{K : a_1, a_2\}, [\beta, \beta]) = \{K_+^{[5]}, K_-^{[5]}\}$.
- If $P_0(\{K : a_1, a_2\}) =$ ,
  set $Q(\{K : a_1, a_2\}, [\alpha, \gamma]) = \{K_+^{[6]}, K_-^{[6]}\}$,
  $Q(\{K : a_1, a_2\}, [\gamma, \alpha]) = \{K_+^{[7]}, K_-^{[7]}\}$,
  and $Q(\{K : a_1, a_2\}, [\alpha, \alpha]) = \{K_+^{[8]}, K_0^{[8]}, K_-^{[8]}\}$.
- If $P_0(\{K_1, K_2 : a_1, a_2\}) =$ ,
  set $Q(\{K_1, K_2 : a_1, a_2\}, [\beta, \beta]) = \{K_+^{[9]}, K_-^{[9]}\}$,
- If $P_0(\{K_1, K_2 : a_1, a_2\}) =$ ,
  set $Q(\{K_1, K_2 : a_1, a_2\}, [\gamma, \alpha]) = \{K^{[10]}\}$,
  $Q(\{K_1, K_2 : a_1, a_2\}, [\alpha, \alpha]) = \{K_+^{[11]}, K_-^{[11]}\}$,
  $Q(\{K_1, K_2 : a_1, a_2\}, [\gamma, \gamma]) = \{K_+^{[12]}, K_-^{[12]}\}$,
  and $Q(\{K_1, K_2 : a_1, a_2\}, [\alpha, \gamma]) = \{K_+^{[13]}, K_0^{[13]}, K_-^{[13]}\}$.

(For convenience, assume $Q(\{K_1, K_2 : a_1, a_2\}, [\gamma, \alpha]))$ is 1-component link.)

- If $P_0(\{K_1, K_2, K_3 : a_1, a_2\}) =$ ,
  set $Q(\{K_1, K_2, K_3 : a_1, a_2\}, [\alpha, \alpha]) = \{K^{[14]}\}$,
  $Q(\{K_1, K_2, K_3 : a_1, a_2\}, [\gamma, \alpha]) = \{K_+^{[15]}, K_-^{[15]}\}$,
  and $Q(\{K_1, K_2, K_3 : a_1, a_2\}, [\alpha, \gamma]) = \{K_+^{[16]}, K_-^{[16]}\}$.
- If $P_0(\{K_1, K_2 : a\}) = \{$   $\}$,
  set $Q(\{K_1, K_2 : a\}, [\alpha]) = \{K_+^{[17]}, K_-^{[17]}, K_0^{[17]}\}$.

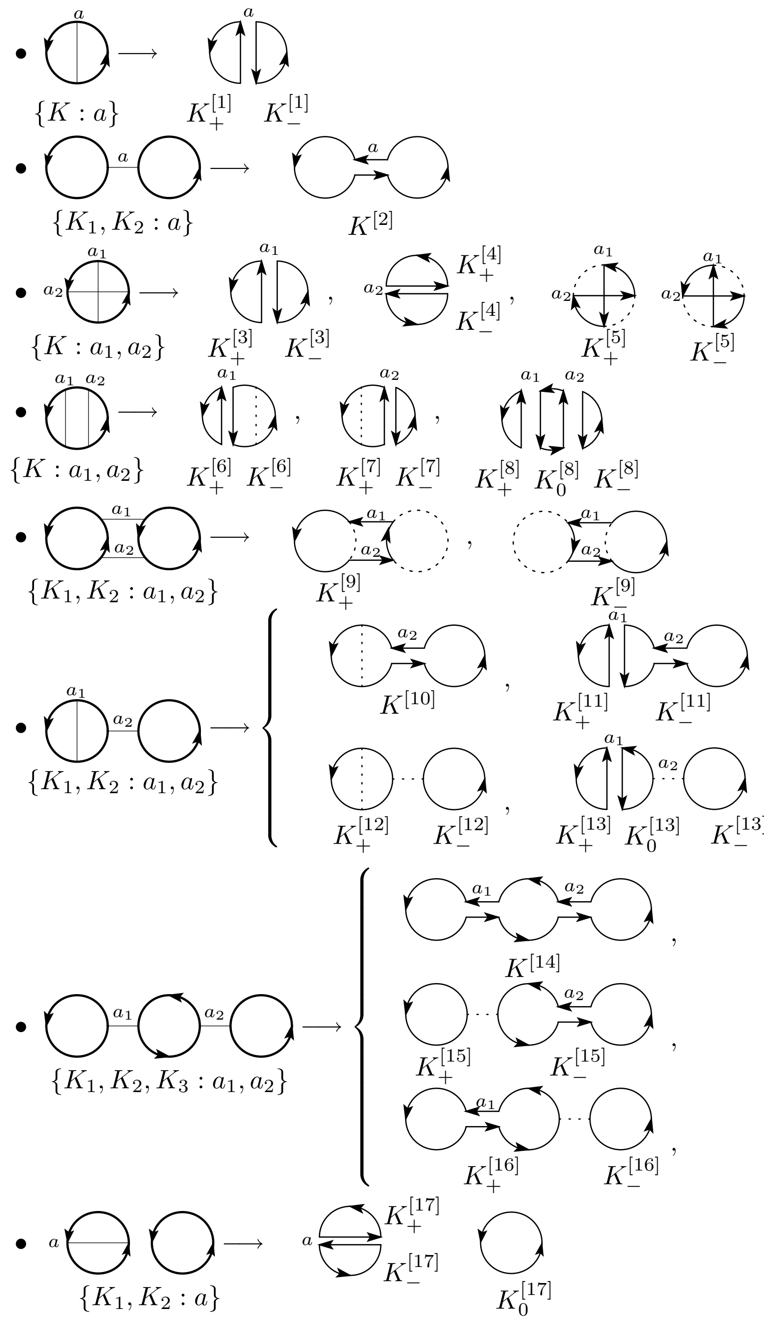


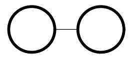
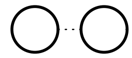
Fig. 14. splitting the crossing to obtain new knot diagrams

5.3. *The main part of the proof of Theorem 2*

This section is the main part of the proof of Theorem 2. The argument is inductive, that is, we shall use the result of v_1 and v_2 for the proof of higher degrees $v_{3.1}, v_{3.2}, v_{4.1}, v_{4.2}, v_{4.3}, v_{4.4}$.

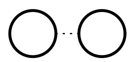
5.3.1. v_1

Proof of (3.1) **in Theorem 2.** We expand (2.7) according to (5.1) and obtain:

$v_1(A^b(\{K_1,K_2\})) = \Big\langle A^b(\{K_1,K_2\}),$  $\Big\rangle + \Big\langle A^b(\{K_1,K_2\}),$  $\Big\rangle.$

The second term in the right side of this equation identically vanishis, since $v_1(A^b(\{K_1,K_2\}))$ and the first term in the right side of this equation are real valued and the second term is pure imaginary. Inserting (5.14) into the first term yields (3.1). □

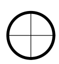
From this proof, we obtain the following lemma.

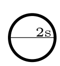
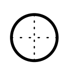
Lemma 5.15. $\Big\langle A^b(\{K_1,K_2\}),$  $\Big\rangle = 0.$ □

Later we shall use this lemma in the proof of higher degrees.

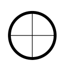
5.3.2. v_2

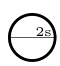
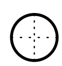
Proof of (3.2) **in Theorem 2.** For sufficiently small b, inserting (5.5) into (2.8) and using (5.15), we have

$v_2(A^b(K)) = \Big\langle G(K),$  $\Big\rangle_\chi$

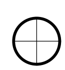
$+\frac{1}{2}\Big\langle G^e(K),$  $\Big\rangle_{\chi^e} + \Big\langle A^b(K),$  $\Big\rangle - \frac{1}{6}m(K) + O(b),$ (5.27)

where we have dropped the imaginary part, since $v_2(A^b(K))$ is real valued. We replace K by $\alpha(K)$ in (5.27) and obtain

$v_2(A^b(\alpha(K))) = \Big\langle G(\alpha(K)),$  $\Big\rangle_\chi$

$+\frac{1}{2}\Big\langle G^e(K),$  $\Big\rangle_{\chi^e} + \Big\langle A^b(K),$  $\Big\rangle - \frac{1}{6}m(K) + O(b).$ (5.28)

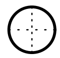
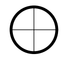
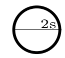
The second, third and fourth terms on the right hand side of (5.27) and (5.28) take the same value for K and $\alpha(K)$, since they are independent on the signatures. Since v_2 is a knot invariant, $v_2(A^b(\alpha(K)))$ is equal to $v_2(U)$, where U denotes a trivial knot. Subtracting (5.28) from (5.27) yields the Gauss diagram formula (3.2) in Theorem 2:

$v_2(A^b(K)) = v_2(U) + \Big\langle \bar{G}(K),$  $\Big\rangle_\chi,$ (5.29)

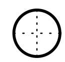
where $\bar{G}(K) = G(K) - G(\alpha(K))$, $v_2(U) = -\frac{1}{6}$ and we have dropped b-dependent term $O(b)$ since $v_2(A^b(K))$ does not depend on b. $\square$

From (5.27) and (5.29), we obtain the following lemma.

Lemma 5.16.

$\Big\langle A^b(K),$  $\Big\rangle = -\Big\langle G(\alpha(K)),$  $\Big\rangle_{\chi} - \frac{1}{2}\Big\langle G^e(K),$  $\Big\rangle_{\chi^e}$

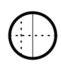
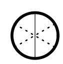
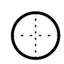
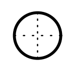
$-\frac{1}{6}(1 - m(K)) + O(b). \quad \square$

Later we shall use this lemma for the proof of higher degrees. Notice that we have obtained the Gauss diagram formula for the difficult integral $\Big\langle A^b(K),$  $\Big\rangle$.

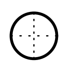
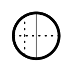
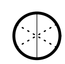
5.3.3. $v_{3.1}$

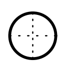
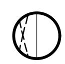
To prepare for the proof of (3.3) in Theorem 2, we shall first prove Lemma 5.17 and Lemma 5.18.

Lemma 5.17.

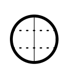
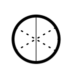
$\Big[\{A^b(K) : a\},$  $+$  $\Big] = \Big\langle A^b(K),$  $\Big\rangle - \sum_{s=\pm}\Big\langle A^b(K_s^{[1]}),$  $\Big\rangle + O(b).$

Proof. Use the identities:

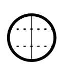
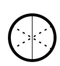
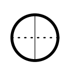
$\Big\langle A^b(K),$  $\Big\rangle = \Big[\{A^b(K) : a\},$  $+$  $+$  $\Big],$

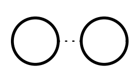
$\sum_{s=\pm}\Big\langle A^b(K_s^{[1]}),$  $\Big\rangle = \Big[\{A^b(K) : a\},$  $\Big] + O(b),$

where $K_s^{[1]}$ are given in Definition 5.14. $\square$

Lemma 5.18. $\Big[\{A^b(K) : a\},$  $+$  $\Big] = O(b).$

Proof.

$\Big[\{A^b(K) : a\},$  $+$  $\Big] = \frac{1}{2}\Big[\{A^b(K) : a\},$  $\Big]^2$

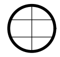
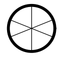
$= \frac{1}{2}\Big\langle A^b(\{K_+^{[1]}, K_-^{[1]}\}),$  $\Big\rangle^2 + O(b)$

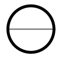
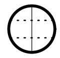
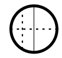
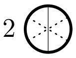
$= O(b)$

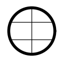
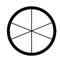
The last step follows from Lemma 5.15. $\square$

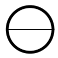
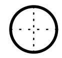
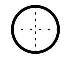
Proof of (3.3) **in Theorem 2.** Inserting (5.6),(5.7) into (2.9) and using (5.21), (5.22),(5.23) yields:

$$v_{3.1}(A^b(K))$$

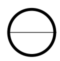
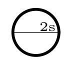
$$= \Big\langle A^b(K),$$  $$+\,2$$  $$\Big\rangle$$

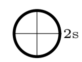
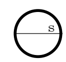
$$+ \sum_a \Big\langle P(\{K:a\}),$$  $$\Big\rangle_\chi \Big[\{A^b(K):a\},$$  $$+$$  $$+\,2$$  $$\Big] + O(b)$$

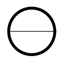
$$= \Big\langle A^b(K),$$  $$+\,2$$  $$\Big\rangle$$

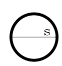
$$+ \sum_a \Big\langle P(\{K:a\}),$$  $$\Big\rangle_\chi \Big\{ \Big\langle A^b(K),$$  $$\Big\rangle - \sum_{s=\pm} \Big\langle A^b(K_s^{[1]}),$$  $$\Big\rangle \Big\} + O(b).$$

The last step follows from Lemma 5.17 and Lemma 5.18. We insert Lemma 5.16 and (5.16) (5.17) into this and use

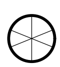
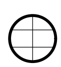
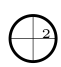
- $\sum_a \Big\langle P(\{K:a\}),$  $\Big\rangle_\chi \Big\langle G^e(K) - \sum_{s=\pm} G(K_s^{[1]}),$  $\Big\rangle_{\chi^e}$

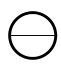
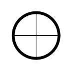
  $= \Big\langle G^e(K),$  $+$  $\Big\rangle_{\chi^e},$

- $\sum_a \Big\langle P(\{K:a\}),$  $\Big\rangle_\chi \Big\{ \big(1-m(K)\big) - \sum_{s=\pm} \big(1-m(K_s^{[1]})\big) \Big\}$

  $= -\Big\langle G^e(K),$  $\Big\rangle_{\chi^e}.$

Then we have

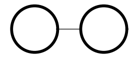
$$v_{3.1}(A^b(K)) = \Big\langle G(K), \quad 2$$  $$+$$  $$+ \frac{1}{2}$$  $$\Big\rangle_\chi$$

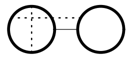
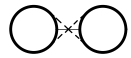
$$- \sum_a \Big\langle P(\{K:a\}),$$  $$\Big\rangle_\chi \Big\langle G(\alpha(K)) - \sum_{s=\pm} G(\alpha(K_s^{[1]})),$$  $$\Big\rangle_\chi,$$

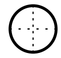
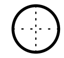
where we have dropped b-dependent term $O(b)$ since $v_{3.1}(A^b(K))$ does not depend on b. We can easily see that this equation is the same as the Gauss diagram formula (3.3) in Theorem 2. □

5.3.4. $v_{3.2}$

To prepare for the proof of (3.4) in Theorem 2, we shall prove Lemma 5.19 and Lemma 5.20.

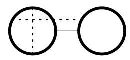
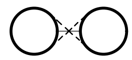
Lemma 5.19. If $P_0(\{K_1, K_2 : a\}) =$ , then

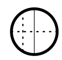
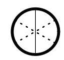
$$\Big[\{A^b(\{K_1, K_2\}) : a\},$$  $+$  $\Big]$

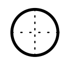
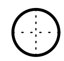
$$= \Big\langle A^b(K^{[2]}),$$  $\Big\rangle - \sum_{i=1}^{2} \Big\langle A^b(K_i),$  $\Big\rangle + O(b).$

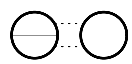
holds.

Proof. The proof is similar to Lemma 5.17 .

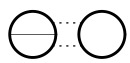
$$\Big[\{A^b(\{K_1, K_2\}) : a\},$$  $+$  $\Big]$

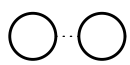
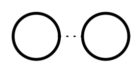
$$= \Big[\{A^b(K^{[2]}) : a\},$$  $+$  $\Big] + O(b)$

$$= \Big\langle A^b(K^{[2]}),$$  $\Big\rangle - \sum_{i=1}^{2} \Big\langle A^b(K_i),$  $\Big\rangle + O(b).$ □

Lemma 5.20. $\Big[\{A^b(\{K_1, K_2\}) : a\},$  $\Big] = O(b).$

Proof.

$$\Big[\{A^b(\{K_1, K_2\}) : a\},$$  $\Big]$

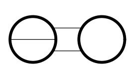
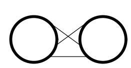
$$= \Big\langle A^b(\{K_+^{[17]}, K_0^{[17]}\}),$$  $\Big\rangle \Big\langle A^b(\{K_-^{[17]}, K_0^{[17]}\}),$  $\Big\rangle + O(b)$

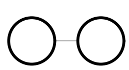
$$= O(b).$$

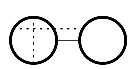
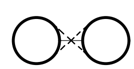
The last step follows from Lemma 5.15. □

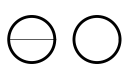
Proof of (3.4) **in Theorem 2.** After inserting (5.8) ,(5.9) into (2.10), we use (5.24),(5.25),(5.26). Then we obtain

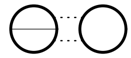
$$v_{3.2}(A^b(\{K_1, K_2\}))$$

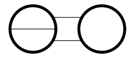
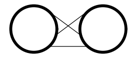
$$= \Big\langle A^b(\{K_1, K_2\}),$$  $+ 2$  $\Big\rangle$

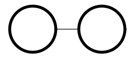
$$+ \sum_a \Big\langle P(K_1, K_2 : a),$$  $\Big\rangle_\chi$

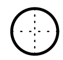
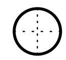
$$\times \Big[\{A^b(\{K_1, K_2\}) : a\},$$  $+$  $\Big]$

$$+ \sum_a \Big\langle P(K_1, K_2 : a),$$  $\Big\rangle_\chi$

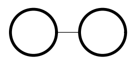
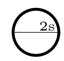
$\times\Big[\{A^b(\{K_1,K_2\}):a\},$  $\Big]+O(b)$

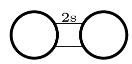
$=\Big\langle A^b(\{K_1,K_2\}),$  $+\,2$  $\Big\rangle$

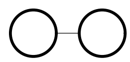
$+\sum_a \Big\langle P(K_1,K_2:a),$  $\Big\rangle_\chi$

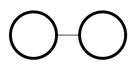
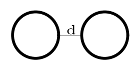
$\times\Big\{\Big\langle A^b(K^{[2]}),$  $\Big\rangle-\sum_{i=1}^{2}\Big\langle A^b(K_i),$  $\Big\rangle\Big\}+O(b).$

The last step follows from Lemma 5.19 and 5.20. We insert Lemma 5.16 and (5.18) (5.19) into this and use

- $\sum_a \Big\langle P(K_1,K_2:a),$  $\Big\rangle_\chi \Big\langle G^e(K^{[2]})-\sum_{i=1,2}G^e(K_i),$  $\Big\rangle_{\chi^e}$

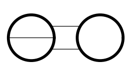
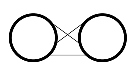
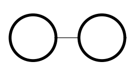
  $=\Big\langle G^e(K_1,K_2),$  $\Big\rangle_{\chi^e},$

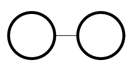
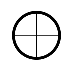
- $\sum_a \Big\langle P(K_1,K_2:a),$  $\Big\rangle_\chi \Big\{\big(1-m(K^{[2]})\big)-\sum_{i=1,2}\big(1-m(K_i)\big)\Big\}$

  $=-\Big\langle G^e(\{K_1,K_2\}),$  $+$  $\Big\rangle_{\chi^e}.$

Then we have

$v_{3.2}(A^b(\{K_1,K_2\}))$

$=\Big\langle G(K_1,K_2),$  $+$  $+\,\frac{1}{3}$  $\Big\rangle_\chi$

$-\sum_a \Big\langle P(K_1,K_2:a),$  $\Big\rangle_\chi \Big\langle G(\alpha(K^{[2]}))-\sum_{i=1,2}G(\alpha(K_i)),$  $\Big\rangle_\chi.$

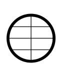
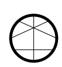
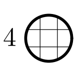
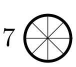
This is the same as the Gauss diagram formula (3.4) in Theorem 2. □

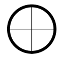
5.3.5. $v_{4.1},v_{4.2},v_{4.3},v_{4.4}$

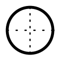
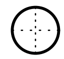
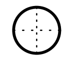
The computaion of degree four $v_{4.1},v_{4.2},v_{4.3},v_{4.4}$ are long but straightforward. We use the same argument as degree two and three.

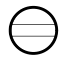
Proof of (3.5) **in Theorem 2.** We calculate $v_{4.1}$ in the same way as the lower degree, and obtain:

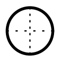
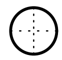
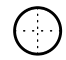
$v_{4.1}(A^b(K))$

$=\Big\langle A^b(K),$  $+$  $+\,2$  $+\,4$  $+\,5$  $+\,7$  $\Big\rangle$

$$+\sum_{(a_1,a_2)}\Big\langle P(K:a_1,a_2),$$  $$\Big\rangle_\chi$$

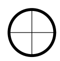
$$\times\Big\{3\Big\langle A^b(K),$$  $$\Big\rangle - 2\sum_{n=3,4}\sum_{s=\pm}\Big\langle A^b(K_s^{[n]}),$$  $$\Big\rangle + \sum_{s=\pm}\Big\langle A^b(K_s^{[5]}),$$  $$\Big\rangle\Big\}$$

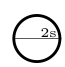
$$+\sum_{(a_1,a_2)}\Big\langle P(K:a_1,a_2),$$  $$\Big\rangle_\chi$$

$$\times\Big\{\Big\langle A^b(K),$$  $$\Big\rangle - \sum_{n=6,7}\sum_{s=\pm}\Big\langle A^b(K_s^{[n]}),$$  $$\Big\rangle + \sum_{s=\pm,0}\Big\langle A^b(K_s^{[8]}),$$  $$\Big\rangle\Big\}$$

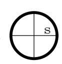
$$+(\text{signature independent terms}) + O(b),$$

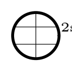
where we have used Lemma 5.15. In the above equation, "(signature independent terms)" means the terms which take the same value for K and $\alpha(K)$. After inserting Lemma 5.16 into this, we use

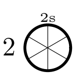
- $$\sum_{(a_1,a_2)}\Big\langle P(K:a_1,a_2),$$  $$\Big\rangle_\chi$$

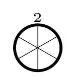
  $$\times\Big\langle 3\,G(K) - 2\sum_{n=3,4}\sum_{s=\pm}G(K_s^{[n]}) + \sum_{s=\pm}G(K_s^{[5]}),$$ 

$$\Big\rangle_\chi$$

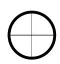
  $$= \Big\langle G(K), 3$$ 

$$+$$ 

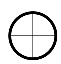
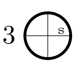
$$+2$$ 

$$+$$ 

$$\Big\rangle_\chi,$$

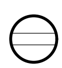
- $$\sum_{(a_1,a_2)}\Big\langle P(K:a_1,a_2),$$  $$\Big\rangle_\chi$$

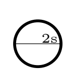
  $$\times\Big\{3\big(1-m(K)\big) - 2\sum_{n=3,4}\sum_{s=\pm}\big(1-m(K_s^{[n]})\big) + \sum_{s=\pm}\big(1-m(K_s^{[5]})\big)\Big\}$$

  $$= \Big\langle G(K),$$  $$-3$$ 

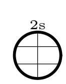
$$\Big\rangle_\chi,$$

and

- $$\sum_{(a_1,a_2)}\Big\langle P(K:a_1,a_2),$$  $$\Big\rangle_\chi$$

  $$\times\Big\langle G(K) - \sum_{n=6,7}\sum_{s=\pm}G(K_s^{[n]}) + \sum_{s=\pm,0}G(K_s^{[8]}),$$ 

$$\Big\rangle_\chi$$

  $$= \Big\langle G(K),$$ 

$$\Big\rangle_\chi,$$

- $$\Big\{\big(1-m(K)\big) - \sum_{n=6,7}\sum_{s=\pm}\big(1-m(K_s^{[n]})\big) + \sum_{s=\pm,0}\big(1-m(K_s^{[8]})\big)\Big\} = 0.$$

Lastly we cancel the signature independent terms by using $\alpha(K)$ in the same argument as in the proof of v_2. Then we have

$$
\begin{aligned}
&v_{4.1}(A^b(K)) \\
&\quad = \frac{1}{360} + \Big\langle \bar{G}(K),\ \ + \ \ + 2 \ \ + 4 \ \ + 5 \ \ + 7 \ \ \Big\rangle_\chi \\
&\qquad\qquad + \Big\langle \bar{G}(K),\ \frac{1}{6} \ \ + \frac{1}{2} \ \ + 2 \ \ + 2 \ \ \Big\rangle_\chi \\
&\quad - \sum_{(a_1,a_2)} \Big\langle \bar{P}(\{K : a_1, a_2\}),\ \ \Big\rangle_\chi \\
&\qquad \times \Big\langle 3\, G(\alpha(K)) - 2 \sum_{n=3,4} \sum_{s=\pm} G(\alpha(K_s^{[n]})) + \sum_{s=\pm} G(\alpha(K_s^{[5]})),\ \ \Big\rangle_\chi \\
&\quad - \sum_{(a_1,a_2)} \Big\langle \bar{P}(\{K : a_1, a_2\}),\ \ \Big\rangle_\chi \\
&\qquad \times \Big\langle G(\alpha(K)) - \sum_{n=6,7} \sum_{s=\pm} G(\alpha(K_s^{[n]})) + \sum_{s=\pm,0} G(\alpha(K_s^{[8]})),\ \ \Big\rangle_\chi .
\end{aligned}
$$

This is the same as the Gauss diagram formula (3.5) in Theorem 2. $\square$

Proof of (3.6) **in Theorem 2.** We calculate $v_{4.2}$ in the same way as the lower degree and obtain:

$$
\begin{aligned}
&v_{4.2}(A^b(K)) \\
&\quad = \Big\langle A^b(K),\ \ + \ \ + \ \ \Big\rangle \\
&\quad + \sum_{(a_1,a_2)} \Big\langle P(K : a_1, a_2),\ \ \Big\rangle_\chi \\
&\quad \times \Big\{ \Big\langle A^b(K),\ \ \Big\rangle - \sum_{n=3,4} \sum_{s=\pm} \Big\langle A^b(K_s^{[n]}),\ \ \Big\rangle + \sum_{s=\pm} \Big\langle A^b(K_s^{[5]}),\ \ \Big\rangle \Big\} \\
&\quad + (\text{signature independent term}) + O(b),
\end{aligned}
$$

where we have used Lemma 5.15. After inserting Lemma 5.16 into this, we use

- $$
\begin{aligned}
&\sum_{(a_1,a_2)} \Big\langle P(K : a_1, a_2),\ \ \Big\rangle_\chi \\
&\qquad \times \Big\langle G(K) - \sum_{n=3,4} \sum_{s=\pm} G(K_s^{[n]}) + \sum_{s=\pm} G(K_s^{[5]}),\ \ \Big\rangle_\chi \\
&\qquad\qquad = \Big\langle G(K),\ \ + \ \ \Big\rangle_\chi ,
\end{aligned}
$$

- $\displaystyle\sum_{(a_1,a_2)} \Big\langle P(K:a_1,a_2),\ \Big\rangle_\chi$

$$\times\Big\{\big(1-m(K)\big) - \sum_{n=3,4}\sum_{s=\pm}\big(1-m(K_s^{[n]})\big) + \sum_{s=\pm}\big(1-m(K_s^{[5]})\big)\Big\}$$

$$= \Big\langle G(K), - \quad + \quad - \quad \Big\rangle_\chi.$$

Lastly we cancel the signature independent terms by using $\alpha(K)$ in the same argument as in the proof of v_2. Then we have

$$v_{4.2}(A^b(K))$$

$$= -\frac{1}{360} + \Big\langle \bar{G}(K), \quad + \quad + \quad + \frac{1}{2} \quad - \frac{1}{6} \quad \Big\rangle_\chi$$

$$- \sum_{(a_1,a_2)} \Big\langle \bar{P}(\{K:a_1,a_2\}),\ \Big\rangle_\chi$$

$$\times\Big\langle G(\alpha(K)) - \sum_{n=3,4}\sum_{s=\pm} G(\alpha(K_s^{[n]})) + \sum_{s=\pm} G(\alpha(K_s^{[5]})),\ \Big\rangle_\chi.$$

This is the same as the Gauss diagram formula (3.6) in Theorem 2. □

Proof of (3.7) **in Theorem 2.** We calculate $v_{4.3}$ in the same way as the lower degree, and we obtain

$$v_{4.3}(A^b(\{K_1,K_2\}))$$

$$= \Big\langle A^b(\{K_1,K_2\}),\ \quad + \quad + 2 \quad + \quad + \quad + \quad \Big\rangle$$

$$+ \sum_{(a_1,a_2)} \Big\langle P(K_1,K_2:a_1,a_2),\ \Big\rangle_\chi$$

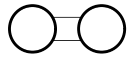
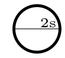
$$\times\Big\{\sum_{i=1,2}\Big\langle A^b(K_i),\ \Big\rangle - \sum_{s=\pm}\Big\langle A^b(K_s^{[9]}),\ \Big\rangle\Big\}$$

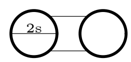
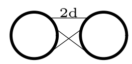
$$+ \sum_{(a_1,a_2)} \Big\langle P(K_1,K_2:a_1,a_2),\ \Big\rangle_\chi$$

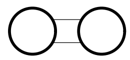
$$\times\Big\{\Big\langle A^b(K^{[10]}),\ \Big\rangle - \sum_{n=11,12}\sum_{s=\pm}\Big\langle A^b(K_s^{[n]}),\ \Big\rangle$$

$$+ \sum_{s=\pm,0}\Big\langle A^b(K_s^{[13]}),\ \Big\rangle\Big\},$$

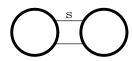
where we have used Lemma 5.15. After inserting Lemma 5.16 into this, we use

- $\sum_{(a_1,a_2)} \Big\langle P(K_1, K_2 : a_1, a_2),$  $\Big\rangle_\chi \Big\langle \sum_{i=1,2} G(K_i) - \sum_{s=\pm} G(K_s^{[9]}),$  $\Big\rangle_\chi$

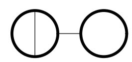
  $= \Big\langle G(\{K_1, K_2\}),$  $-$  $\Big\rangle_\chi,$

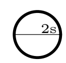
- $\sum_{(a_1,a_2)} \Big\langle P(K_1, K_2 : a_1, a_2),$  $\Big\rangle_\chi$

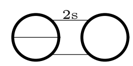
  $\times \Big\{ \sum_{i=1,2} \big(1 - m(K_i)\big) - \sum_{s=\pm} \big(1 - m(K_s^{[9]})\big) \Big\}$

  $= \Big\langle G(\{K_1, K_2\}),$  $\Big\rangle_\chi,$

and

- $\sum_{(a_1,a_2)} \Big\langle P(K_1, K_2 : a_1, a_2),$  $\Big\rangle_\chi$

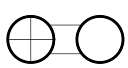
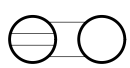
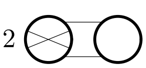
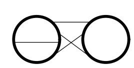
  $\times \Big\langle G(K^{[10]}) - \sum_{n=11,12} \sum_{s=\pm} G(K_s^{[n]}) + \sum_{s=\pm,0} G(K_s^{[13]}),$  $\Big\rangle_\chi$

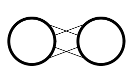
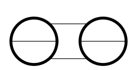
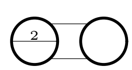
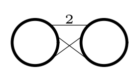
  $= \Big\langle G(\{K_1, K_2\}),$  $\Big\rangle_\chi,$

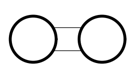
- $\big(1 - m(K^{[10]})\big) - \sum_{n=11,12} \sum_{s=\pm} \big(1 - m(K_s^{[n]})\big) + \sum_{s=\pm,0} \big(1 - m(K_s^{[13]})\big) = 0.$

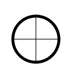
Lastly we cancel the signature independent terms by using $\alpha(K)$ in the same argument as in the proof of v_2. Then we have

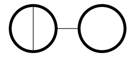
$v_{4.3}(A^b(\{K_1, K_2\}))$

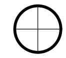
$= \Big\langle \bar{G}(\{K_1, K_2\}),$  $+$  $+ 2$  $+$  $\Big\rangle_\chi$

$+ \Big\langle \bar{G}(\{K_1, K_2\}),$  $+$  $+ \frac{1}{2}$  $+ \frac{1}{2}$  $\Big\rangle_\chi$

$- \sum_{(a_1,a_2)} \Big\langle \bar{P}(\{K_1, K_2 : a_1, a_2\}),$  $\Big\rangle_\chi$

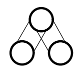
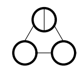
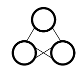
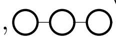
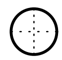
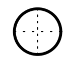
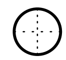
$\times \Big\langle \sum_{i=1,2} G(\alpha(K_i)) - \sum_{s=\pm} G(\alpha(K_s^{[9]})),$  $\Big\rangle_\chi$

$- \sum_{(a_1,a_2)} \Big\langle \bar{P}(\{K_1, K_2 : a_1, a_2\}),$  $\Big\rangle_\chi$

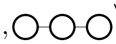
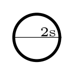
$$\times\Big\langle G(\alpha(K^{[10]})) - \sum_{n=11,12}\sum_{s=\pm} G(\alpha(K_s^{[n]})) + \sum_{s=\pm,0} G(\alpha(K_s^{[13]})),$$

$$\Big\rangle_\chi$$

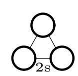
This is the same as the Gauss diagram formula (3.7) in Theorem 2. $\square$

Proof of (3.8) **in Theorem 2.** We calculate $v_{4.4}$ in the same way as the lower degree, and we obtain

$$v_{4.4}(A^b(\{K_1,K_2,K_3\}))$$
$$= \Big\langle A^b(\{K_1,K_2,K_3\}),$$

$$+$$

$$+$$

$$\Big\rangle$$
$$+ \sum_{(a_1,a_2)} \Big\langle P(K_1,K_2,K_3 : a_1,a_2),$$

$$\Big\rangle_\chi$$
$$\times\Big\{ \Big\langle A^b(K^{[14]}),$$

$$\Big\rangle - \sum_{n=15,16}\sum_{s=\pm} \Big\langle A^b(K_s^{[n]}),$$

$$\Big\rangle$$
$$+ \sum_{i=1,2,3} \Big\langle A^b(K_i),$$

$$\Big\rangle \Big\},$$

where we have used Lemma 5.15. After inserting Lemma 5.16 into this, we use

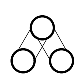
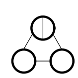
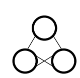
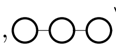
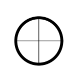
- $$\sum_{(a_1,a_2)} \Big\langle P(K_1,K_2,K_3 : a_1,a_2),$$

$$\Big\rangle_\chi$$
$$\times\Big\langle G(K^{[14]}) - \sum_{n=15,16}\sum_{s=\pm} G(K_s^{[n]}) + \sum_{i=1,2,3} G(K_i),$$


$$\Big\rangle_\chi$$
$$= \Big\langle G(\{K_1,K_2,K_3\}),$$


$$\Big\rangle_\chi,$$

- $$\big(1-m(K^{[14]})\big) - \sum_{n=15,16}\sum_{s=\pm}\big(1-m(K_s^{[n]})\big) + \sum_{i=1,2,3}\big(1-m(K_i)\big) = 0.$$

Lastly we cancel the signature independent terms by using $\alpha(K)$ in the same argument as in the proof of v_2. Then we have

$$v_{4.4}(A^b(\{K_1,K_2,K_3\}))$$
$$= \Big\langle \bar{G}(\{K_1,K_2,K_3\}),$$

$$+$$

$$+$$

$$\Big\rangle_\chi$$
$$- \sum_{(a_1,a_2)} \Big\langle \bar{P}(\{K_1,K_2,K_3 : a_1,a_2\}),$$

$$\Big\rangle_\chi$$
$$\times\Big\langle G(\alpha(K^{[14]})) - \sum_{n=15,16}\sum_{s=\pm} G(\alpha(K_s^{[n]})) + \sum_{i=1}^{3} G(\alpha(K_i)),$$

$$\Big\rangle_\chi.$$

This is the same as the Gauss diagram formula (3.8) in Theorem 2. □

Remark. Notice the concept of direction "s", "d" disappear in the final Gauss diagram formula, as we have expected.

6. Consistency Check

We write $\hat{P}_L^{(4)}$ for the right hand side of (4.3):

$$\hat{P}_L^{(4)} = W_{su(N)}^{(4)}\left(N^{n-1}\left\{\exp\Big(\sum_{D\in\mathfrak{D}_K} D\ u(D:L)\Big)\right\}\left\{\sum_{D\in\mathfrak{D}_L} D\ w(D:L)\right\}\right). \quad (6.1)$$

From Corollary 1, it is trivial that $\hat{P}_L^{(4)}$ satisfies the Homfly skein relation (4.1) up to degree four. But as a consitency check of the Gauss diagram formula, we will prove directly that $\hat{P}_L^{(4)}$ really satisfies the HOMFLY skein relation up to degree four *by using the Gauss diagram formula in Theorem* 2:

$$\left[\exp(\frac{Nx}{2})\hat{P}_{L_+}^{(4)} - \exp(-\frac{Nx}{2})\hat{P}_{L_-}^{(4)} - (e^{\frac{x}{2}} - e^{-\frac{x}{2}})\hat{P}_{L_0}^{(4)}\ \right]^{(4)} = 0. \quad (6.2)$$

Proof. There are two cases:

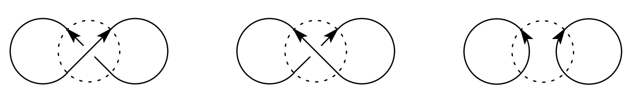
(1) L_+ and L_- have n-component, while L_0 has $(n+1)$-components.

(2) L_+ and L_- have $(n+1)$-component, while L_0 has n-components.

First, we consider the case (1). Set

$$\begin{aligned} L_+ &= \{K_1, \cdots, K_{n-1}, K_n^+\} \\ L_- &= \{K_1, \cdots, K_{n-1}, K_n^-\} \\ L_0 &= \{K_1, \cdots, K_{n-1}, K_n^0, K_{n+1}^0\}. \end{aligned}$$

The $(n-1)$-components $K_1, \cdots, K_{n-1}$ are common in L_+, L_-, L_0, while K_n^+, K_n^-, K_n^0, K_{n+1}^0 are the same except inside the dashed circle:



K_n^+ K_n^- K_n^0 K_{n+1}^0

Inserting (6.1) into the left side of (6.2) and using Appendix D, we have

$$\begin{aligned} &\left[\exp(\frac{Nx}{2})\hat{P}_{L_+}^{(4)} - \exp(-\frac{Nx}{2})\hat{P}_{L_-}^{(4)} - (e^{\frac{x}{2}} - e^{-\frac{x}{2}})\hat{P}_{L_0}^{(4)}\ \right]^{(4)} \\ &= -\frac{1}{4}(N^2-1)x^2\ V_1 + \frac{1}{8}N(N^2-1)x^3\ V_2 - \frac{(N^2-1)}{8N}x^3\ V_3 - \frac{N^2(N^2-1)}{16}x^4\ V_4 \\ &\qquad + \frac{(N^2-1)(N^2+2)}{16}x^4\ V_5 + \frac{(N^2-1)}{16}x^4\ V_6 - \frac{(N^2-1)}{16N^2}x^4\ V_7, \end{aligned}$$

where,

- $V_1 = \Big\{\sum_{s=\pm} s\ v_2(K_n^s)\Big\} - 2v_1\big(\{K_n^0, K_{n+1}^0\}\big)$,

- $V_2 = \Big\{\sum_{s=\pm} s\ v_{3.1}(K_n^s)\Big\} - \Big\{\sum_{s=\pm} v_2(K_n^s)\Big\} + 2\Big\{\sum_{i=n}^{n+1} v_2(K_i^0)\Big\}$
$$-\Big\{v_1(K_n^0, K_{n+1}^0)\Big\}^2 + \frac{1}{3},$$

- $V_3 = \sum_{i=1}^{n-1}\Big[\Big\{\sum_{s=\pm} s\ v_{3.2}\big(\{K_i, K_n^s\}\big)\Big\} - 2v_1(K_i, K_n^0)v_1\big(\{K_i, K_{n+1}^0\}\big)\Big]$,

- $V_4 = \Big\{\sum_{s=\pm} s\ v_{4.1}(K_n^s)\Big\} - \Big\{\sum_{s=\pm} v_{3.1}(K_n^s)\Big\} + 2\Big\{\sum_{i=n}^{n+1} v_{3.1}(K_i^0)\Big\}$
$$+v_{3.2}\big(\{K_n^0, K_{n+1}^0\}\big) - \frac{3}{2}v_1\big(\{K_n^0, K_{n+1}^0\}\big)\Big\{\sum_{s=\pm} v_2(K_n^s) - 2\sum_{i=n}^{n+1} v_2(K_i^0)\Big\}$$
$$-\frac{1}{3}\Big\{v_1(K_n^0, K_{n+1}^0)\Big\}^3 + \frac{7}{6}v_1\big(\{K_n^0, K_{n+1}^0\}\big),$$

- $V_5 = \Big\{\sum_{s=\pm} s\ v_{4.2}(K_n^s)\Big\} + v_{3.2}\big(\{K_n^0, K_{n+1}^0\}\big) + \frac{1}{6}v_1\big(\{K_n^0, K_{n+1}^0\}\big)$
$$-\frac{1}{2}v_1\big(\{K_n^0, K_{n+1}^0\}\big)\Big\{\sum_{s=\pm} v_2(K_n^s) - 2\sum_{i=n}^{n+1} v_2(K_i^0)\Big\},$$

- $V_6 = \sum_{i=1}^{n-1}\Big[\Big\{\sum_{s=\pm} s\ v_{4.3}\big(\{K_i, K_n^s\}\big)\Big\} - \Big\{\sum_{s=\pm} v_{3.2}\big(\{K_i, K_n^s\}\big)\Big\}$
$$+2\Big\{\sum_{j=n}^{n+1} v_{3.2}\big(\{K_i, K_j^0\}\big)\Big\} - 2v_1(K_n^0, K_{n+1}^0)v_1\big(\{K_i, K_n^0\}\big)v_1\big(\{K_i, K_{n+1}^0\}\big)\Big],$$

- $V_7 = \sum_{1\leq i<j\leq n-1}\Big[\Big\{\sum_{s=\pm} s\ v_{4.4}\big(\{K_i, K_j, K_n^s\}\big)\Big\}$
$$-2v_1\big(\{K_i, K_j\}\big)\Big\{v_1\big(\{K_i, K_n^0\}\big)v_1\big(\{K_j, K_{n+1}^0\}\big)$$
$$+v_1\big(\{K_i, K_{n+1}^0\}\big)v_1\big(\{K_j, K_n^0\}\big)\Big\}\Big].$$

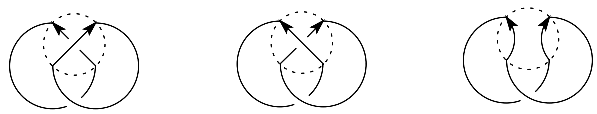
Inserting the Guass diagram formula (Theorem 2) into each V_i, we find out all of these equations vanishes identically $V_i = 0$ $(i = 1, \cdots, 7)$. This shows the skein relation (6.2) holds in case of (1).

Next we consider the case (2). Set

$$\begin{aligned} L_+ &= \{K_1, \cdots, K_{n-1}, K_n^+, K_{n+1}^+\} \\ L_- &= \{K_1, \cdots, K_{n-1}, K_n^-, K_{n+1}^-\} \end{aligned}$$

$$L_0 \quad = \quad \{K_1, \cdots, K_{n-1}, K_n^0\}.$$

The $(n-1)$-components $K_1, \cdots, K_{n-1}$ are common in L_+, L_-, L_0, while K_n^+, K_{n+1}^+, K_n^-, K_{n+1}^-, K_n^0 are the same except inside the dashed circle:



Inserting (6.1) into the left side of (6.2) and using Appendix D, we have

$$\Big[\exp(\frac{Nx}{2})\hat{P}_{L_+}^{(4)} - \exp(-\frac{Nx}{2})\hat{P}_{L_-}^{(4)} - (e^{\frac{x}{2}} - e^{-\frac{x}{2}})\hat{P}_{L_0}^{(4)}\ \Big]^{(4)}$$
$$= \quad -\frac{(N^2-1)}{8}x^3\ V_8 + \frac{N(N^2-1)}{16}x^4\ V_9 - \frac{(N^2-1)}{16N}x^4\ V_{10},$$

where,

- $V_8 = \Big\{\sum_{s=\pm} s\ v_{3.2}\big(\{K_n^s, K_{n+1}^s\}\big)\Big\} + 2\Big\{\sum_{i=n}^{n+1} v_2(K_i^+)\Big\} - 2v_2(K_n^0) - \frac{1}{3},$
- $V_9 = \Big\{\sum_{s=\pm} s\ v_{4.3}\big(\{K_n^s, K_{n+1}^s\}\big)\Big\} + 2\Big\{\sum_{i=n}^{n+1} v_{3.1}(K_i^+)\Big\} - 2v_{3.1}(K_n^0)$
  $+\frac{p}{2}\Big\{\sum_{s=\pm} s\ v_{3.2}\big(\{K_n^s, K_{n+1}^s\}\big)\Big\} - \frac{1}{2}\Big\{\sum_{s=\pm} v_{3.2}\big(\{K_n^s, K_{n+1}^s\}\big)\Big\},$
- $V_{10} = \sum_{i=1}^{n-1}\Big[\Big\{\sum_{s=\pm} s\ v_{4.4}\big(\{K_i, K_n^s, K_{n+1}^s\}\big)\Big\} + 2\Big\{\sum_{j=n}^{n+1} v_{3.2}\big(\{K_i, K_j^+\}\big)\Big\}$
  $-2v_{3.2}\big(\{K_i, K_n^0\}\big)\Big],$

where $p = v_1\big(\{K_n^+, K_{n+1}^+\}\big) - 1$. Inserting the Guass diagram formula (Theorem 2) into each V_i, we find out all of these equations vanishes identically $V_i = 0$ ($i = 8, 9, 10$). This shows the skein relation (6.2) holds in case of (2). $\square$

Acknowledgements

The author would like to express his hearty thanks to his thesis advisor Professor T.Eguchi for his advise and constant encouragement, and also to Professor T.Kohno for his kind suggestion for further discussion.

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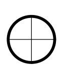
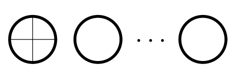
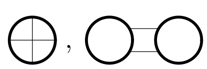
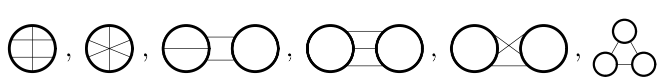
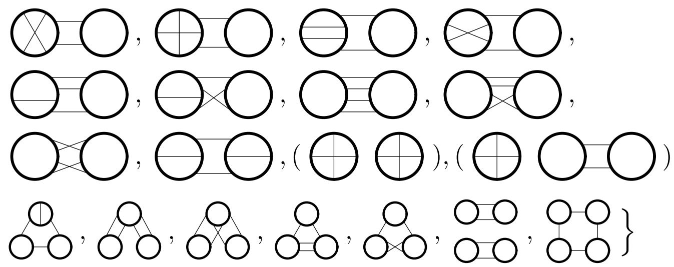
Appendix A

We list the chord diagrams of n-circles without any isolated chord up to degree four (see the proof of Theorem 1).

For convenience, we omit the circles which have no chord. For example, we write

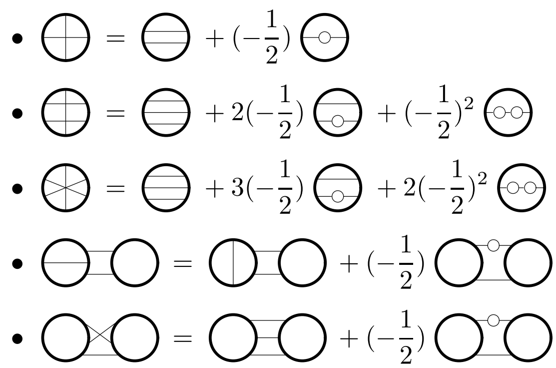
() instead of () , etc.

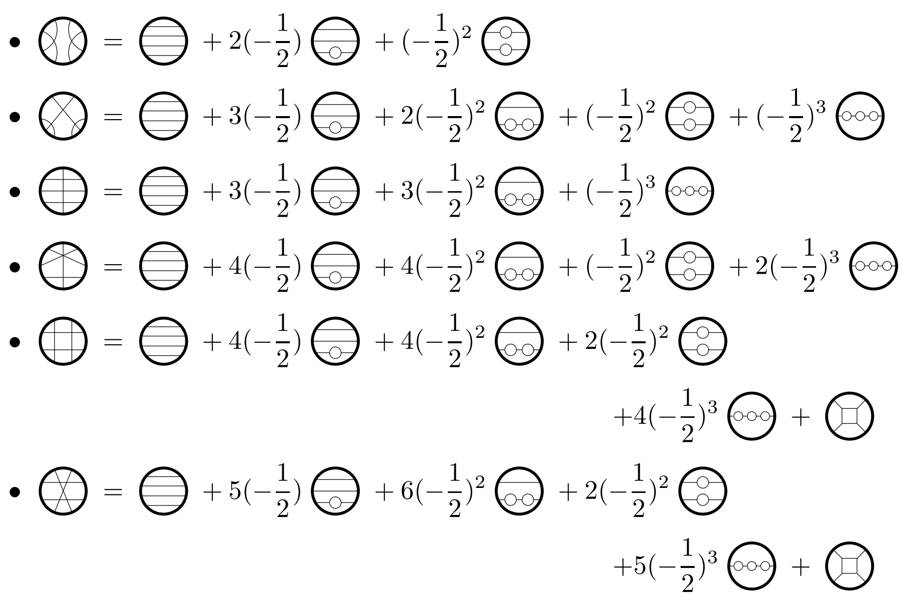
- $\bar{\mathfrak{D}}_2 = \Big\{$  $\Big\}$
- $\bar{\mathfrak{D}}_3 = \Big\{$  $\Big\}$
- $\bar{\mathfrak{D}}_4 = \Big\{$ 

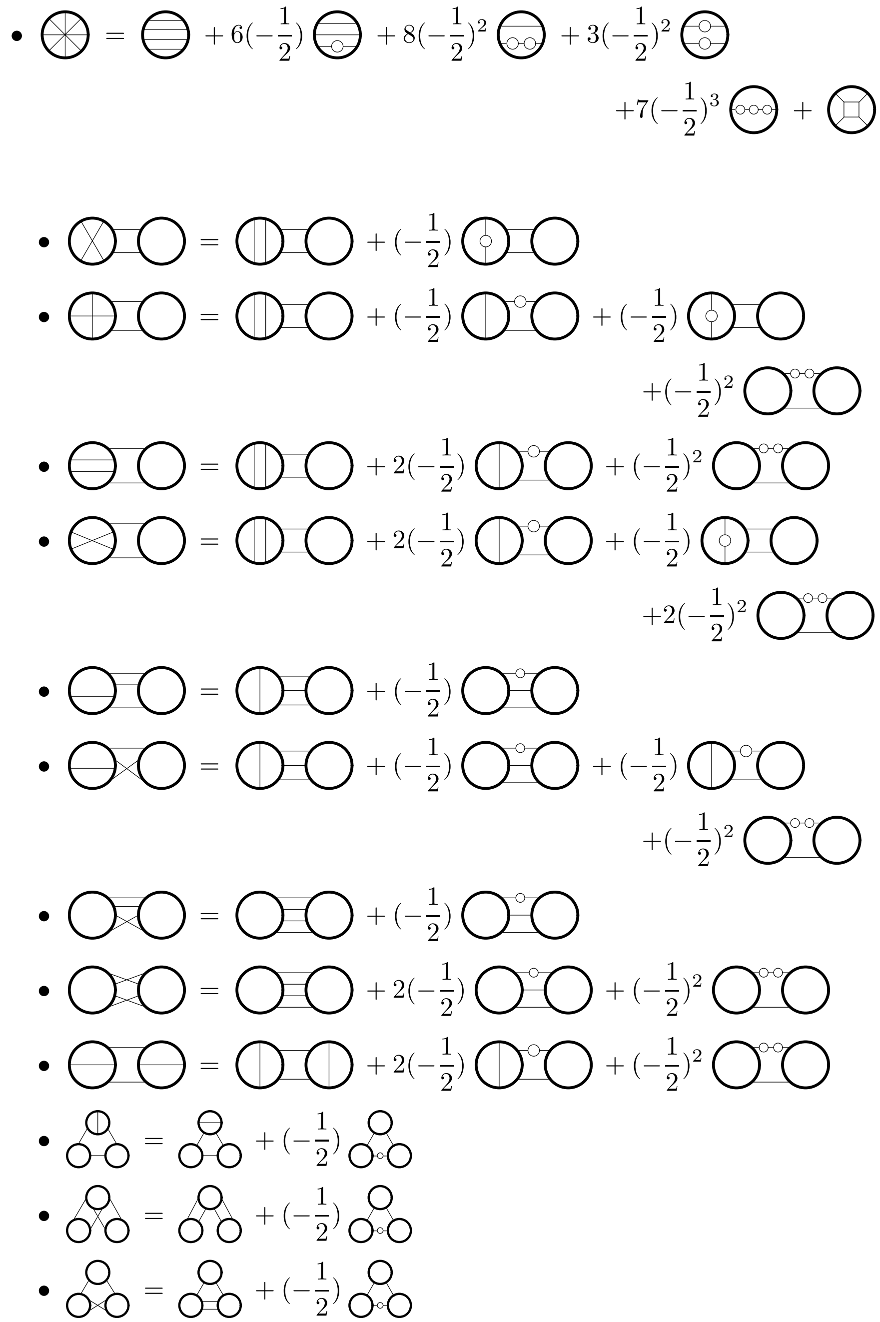


Appendix B

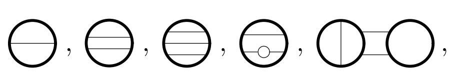
We expand chord diagrams into CC diagrams, using AS, IHX, STU relation as follows (see the proof of Theorem 1).

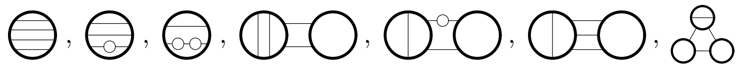






Notice we set the diagrams

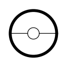
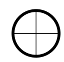


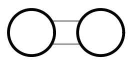
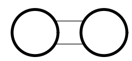
, etc,

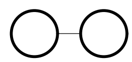
to be 0 by framing independence.

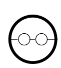
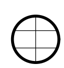
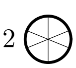
Appendix C

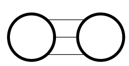
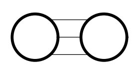
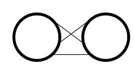
We compute each cofficient of the CC diagram in (2.13) (see the proof of Theorem 1).

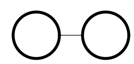
- $\Big($the cofficient of $(-\frac{1}{2})$  $\Big) = \sum_{i=1}^{n} \Big\langle\!\!\Big\langle \mathbf{K}_i,$  $\Big\rangle\!\!\Big\rangle$,

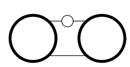
- $\Big($the cofficient of  $\Big) = \sum_{i<j} \Big\langle\!\!\Big\langle \{\mathbf{K}_i, \mathbf{K}_j\},$  $\Big\rangle\!\!\Big\rangle$

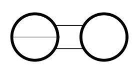
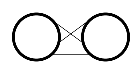
  $= \sum_{i<j} \frac{1}{2} \Big\langle\!\!\Big\langle \{\mathbf{K}_i, \mathbf{K}_j\},$  $\Big\rangle\!\!\Big\rangle^2$,

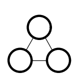
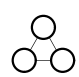
- $\Big($the cofficient of $(-\frac{1}{2})^2$  $\Big) = \sum_{i=1}^{n} \Big\langle\!\!\Big\langle \mathbf{K}_i,$  $+\,2$  $\Big\rangle\!\!\Big\rangle$,

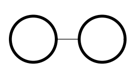
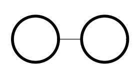
- $\Big($the cofficient of  $\Big) = \sum_{i<j} \Big\langle\!\!\Big\langle \{\mathbf{K}_i, \mathbf{K}_j\},$  $+$  $\Big\rangle\!\!\Big\rangle$

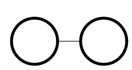
  $= \sum_{i<j} \frac{1}{3!} \Big\langle\!\!\Big\langle \{\mathbf{K}_i, \mathbf{K}_j\},$  $\Big\rangle\!\!\Big\rangle^3$,

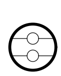
- $\Big($the cofficient of $(-\frac{1}{2})$  $\Big)$

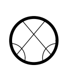
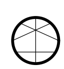
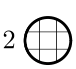
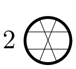
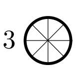
  $= \sum_{i<j} \Big\langle\!\!\Big\langle \{\mathbf{K}_i, \mathbf{K}_j\},$  $+$  $\Big\rangle\!\!\Big\rangle$,

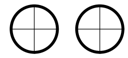
- $\Big($the cofficient of  $\Big) = \sum_{i<j<k} \Big\langle\!\!\Big\langle \{\mathbf{K}_i, \mathbf{K}_j, \mathbf{K}_k\},$  $\Big\rangle\!\!\Big\rangle$

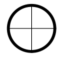
  $= \sum_{1\le i<j<k\le n} \Big\langle\!\!\Big\langle \{\mathbf{K}_i, \mathbf{K}_j\},$  $\Big\rangle\!\!\Big\rangle \Big\langle\!\!\Big\langle \{\mathbf{K}_j, \mathbf{K}_k\},$  $\Big\rangle\!\!\Big\rangle$

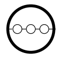
  $\times \Big\langle\!\!\Big\langle \{\mathbf{K}_k, \mathbf{K}_i\},$  $\Big\rangle\!\!\Big\rangle$

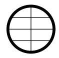
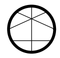
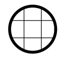
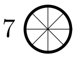
- $\Big($the cofficient of $(-\frac{1}{2})^2$  $\Big)$

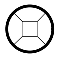
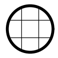
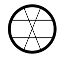
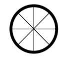
  $= \sum_{i=1}^{n} \Big\langle\!\!\Big\langle \mathbf{K}_i,$  $+$  $+$  $+\,2$  $+\,2$  $+\,3$  $\Big\rangle\!\!\Big\rangle$

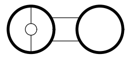
$$+\sum_{i<j}\Big\langle\!\!\Big\langle \{\mathbf{K}_i,\mathbf{K}_j\},\{$$  $$\}\Big\rangle\!\!\Big\rangle$$

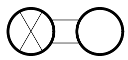
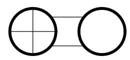
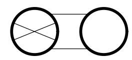
$$=\frac{1}{2}\Big\{\sum_{i=1}^{n}\Big\langle\!\!\Big\langle \mathbf{K}_i,$$  $$\Big\rangle\!\!\Big\rangle\Big\}^2,$$

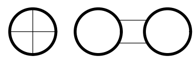
• (the cofficient of $(-\frac{1}{2})^3$ )

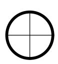
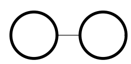
$$=\sum_{i=1}^{n}\Big\langle\!\!\Big\langle \mathbf{K}_i,$$  $+$  $+2$  $+4$  $+5$  $+7$  $$\Big\rangle\!\!\Big\rangle,$$

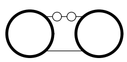
• (the cofficient of ) $=\sum_{i=1}^{n}\Big\langle\!\!\Big\langle \mathbf{K}_i,$  $+$  $+$  $\Big\rangle\!\!\Big\rangle,$

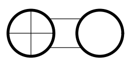
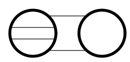
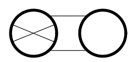
• (the cofficient of $(-\frac{1}{2})$ )

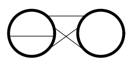
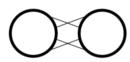
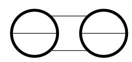
$$=\sum_{i<j}\Big\langle\!\!\Big\langle \{\mathbf{K}_i,\mathbf{K}_j\},$$  $+$  $+$  $$\Big\rangle\!\!\Big\rangle$$

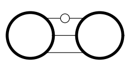
$$+\sum_{i<j<k}\Big\langle\!\!\Big\langle \{\mathbf{K}_i,\mathbf{K}_j,\mathbf{K}_k\},\{$$  $$\}\Big\rangle\!\!\Big\rangle$$

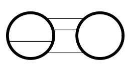
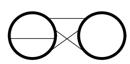
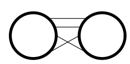
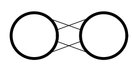
$$=\Big\{\sum_{i=1}^{n}\Big\langle\!\!\Big\langle \mathbf{K}_i,$$  $$\Big\rangle\!\!\Big\rangle\Big\}\Big\{\sum_{i<j}\frac{1}{2}\Big\langle\!\!\Big\langle \{\mathbf{K}_i,\mathbf{K}_j\},$$  $$\Big\rangle\!\!\Big\rangle\Big\}^2,$$

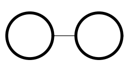
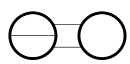
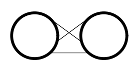
• (the cofficient of $(-\frac{1}{2})^2$ )

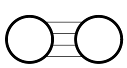
$$=\sum_{i<j}\Big\langle\!\!\Big\langle \{\mathbf{K}_i,\mathbf{K}_j\},$$  $+$  $+2$ 

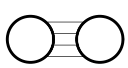
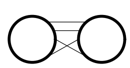
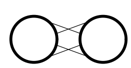
$+$  $+$  $+$  $\Big\rangle\!\!\Big\rangle,$

• (the cofficient of $(-\frac{1}{2})$ )

$$=\sum_{i<j}\Big\langle\!\!\Big\langle \{\mathbf{K}_i,\mathbf{K}_j\},$$  $+$  $+$  $+2$  $$\Big\rangle\!\!\Big\rangle$$

$$=\sum_{i<j}\frac{1}{2}\Big\langle\!\!\Big\langle \{\mathbf{K}_i,\mathbf{K}_j\},$$  $$\Big\rangle\!\!\Big\rangle\Big\langle\!\!\Big\langle \{\mathbf{K}_i,\mathbf{K}_j\},$$  $+$  $$\Big\rangle\!\!\Big\rangle,$$

• (the cofficient of )

$$=\sum_{i<j}\Big\langle\!\!\Big\langle \{\mathbf{K}_i,\mathbf{K}_j\},$$  $+$  $+$  $$\Big\rangle\!\!\Big\rangle$$

$$= \sum_{i<j} \frac{1}{4!} \left\langle\!\!\left\langle \{\mathbf{K}_i, \mathbf{K}_j\}, \quad \right\rangle\!\!\right\rangle^4,$$

$$\bullet\Big(\text{the cofficient of } (-\frac{1}{2}) \quad \Big)$$

$$= \sum_{i<j<k} \left\langle\!\!\left\langle \{\mathbf{K}_i, \mathbf{K}_j.\mathbf{K}_k\}, \quad + \quad + \quad \right\rangle\!\!\right\rangle$$

$$\bullet\Big(\text{the cofficient of } \quad \Big) = \sum_{i<j<k} \left\langle\!\!\left\langle \{\mathbf{K}_i, \mathbf{K}_j.\mathbf{K}_k\}, \quad + \quad \right\rangle\!\!\right\rangle$$

$$= \sum_{\substack{1\le i<j<k\le n \\ 1\le j<i<k\le n \\ 1\le j<k<i\le n}} \frac{1}{2} \left\langle\!\!\left\langle \{\mathbf{K}_i, \mathbf{K}_j\}, \quad \right\rangle\!\!\right\rangle^2 \frac{1}{2} \left\langle\!\!\left\langle \{\mathbf{K}_i, \mathbf{K}_k\}, \quad \right\rangle\!\!\right\rangle^2$$

$$\bullet\Big(\text{the cofficient of } \quad \Big) = \sum_{i<j<k} \left\langle\!\!\left\langle \{\mathbf{K}_i, \mathbf{K}_j.\mathbf{K}_k\}, \quad + \quad \right\rangle\!\!\right\rangle$$

$$= \sum_{\substack{1\le i<j<k\le n \\ 1\le j<i<k\le n \\ 1\le j<k<i\le n}} \left\langle\!\!\left\langle \{\mathbf{K}_i, \mathbf{K}_j\}, \quad \right\rangle\!\!\right\rangle \left\langle\!\!\left\langle \{\mathbf{K}_i, \mathbf{K}_k\}, \quad \right\rangle\!\!\right\rangle \times \frac{1}{2} \left\langle\!\!\left\langle \{\mathbf{K}_j, \mathbf{K}_k\}, \quad \right\rangle\!\!\right\rangle^2$$

$$\bullet\Big(\text{the cofficient of } \quad \Big) = \sum_{i<j<k<l} \left\langle\!\!\left\langle \{\mathbf{K}_i, \mathbf{K}_j.\mathbf{K}_k, \mathbf{K}_l\}, \quad \right\rangle\!\!\right\rangle$$

$$= \sum_{\substack{1\le i<j<k<l\le n \\ 1\le i<k<j<l\le n \\ 1\le i<k<l<j\le n}} \frac{1}{2} \left\langle\!\!\left\langle \{\mathbf{K}_i, \mathbf{K}_j\}, \quad \right\rangle\!\!\right\rangle^2 \frac{1}{2} \left\langle\!\!\left\langle \{\mathbf{K}_k, \mathbf{K}_l\}, \quad \right\rangle\!\!\right\rangle^2$$

$$\bullet\Big(\text{the cofficient of } \quad \Big) = \sum_{i<j<k<l} \left\langle\!\!\left\langle \{\mathbf{K}_i, \mathbf{K}_j.\mathbf{K}_k, \mathbf{K}_l\}, \quad \right\rangle\!\!\right\rangle$$

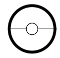
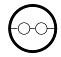
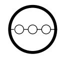
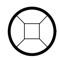
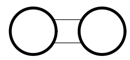
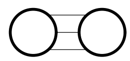
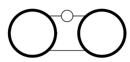
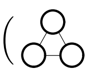
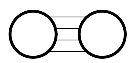
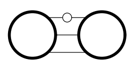
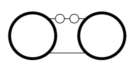
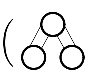
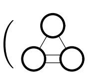
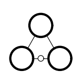
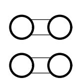
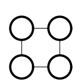
$$= \sum_{\substack{1\le i<k<j<l\le n \\ 1\le i<j<k<l\le n \\ 1\le i<j<l<k\le n}} \left\langle\!\!\left\langle \{\mathbf{K}_i, \mathbf{K}_j\}, \quad \right\rangle\!\!\right\rangle \left\langle\!\!\left\langle \{\mathbf{K}_j, \mathbf{K}_k\}, \quad \right\rangle\!\!\right\rangle$$

$$\times \left\langle\!\!\left\langle \{\mathbf{K}_k, \mathbf{K}_l\}, \quad \right\rangle\!\!\right\rangle \left\langle\!\!\left\langle \{\mathbf{K}_l, \mathbf{K}_i\}, \quad \right\rangle\!\!\right\rangle.$$

Appendix D

We list the acutual table of the weight system. The computaion is straightfor-

ward, using Definition 2.4.

- $W_{su(N)}\Big(\big(-\frac{1}{2}\big)$  $\Big) = -x^2\,\frac{N^2-1}{4}$,
- $W_{su(N)}\Big(\big(-\frac{1}{2}\big)^2$  $\Big) = x^3\,\frac{N(N^2-1)}{8}$,
- $W_{su(N)}\Big(\big(-\frac{1}{2}\big)^3$  $\Big) = -x^4\,\frac{N^2(N^2-1)}{16}$,
- $W_{su(N)}\Big($  $\Big) = x^4\,\frac{(N^2-1)(N^2+2)}{16}$,
- $W_{su(N)}\Big($  $\Big) = x^2\,\frac{(N^2-1)}{4N^2}$,
- $W_{su(N)}\Big($  $\Big) = x^3\,\frac{(N^2-1)(N^2-2)}{8N^3}$,
- $W_{su(N)}\Big(\big(-\frac{1}{2}\big)$  $\Big) = -x^3\,\frac{N^2-1}{8N}$,
- $W_{su(N)}\Big($  $\Big) = x^3\,\frac{N^2-1}{8N^3}$,
- $W_{su(N)}\Big($  $\Big) = x^4\,\frac{(N^2-1)(N^4-3N^2+3)}{16N^4}$,
- $W_{su(N)}\Big(\big(-\frac{1}{2}\big)$  $\Big) = -x^4\,\frac{(N^2-1)(N^2-2)}{16N^2}$,
- $W_{su(N)}\Big(\big(-\frac{1}{2}\big)^2$  $\Big) = x^4\,\frac{(N^2-1)}{16}$,
- $W_{su(N)}\Big($  $\Big) = x^4\,\frac{(N^2-1)^2}{16N^4}$,
- $W_{su(N)}\Big($  $\Big) = x^4\,\frac{(N^2-1)(N^2-2)}{16N^4}$,
- $W_{su(N)}\Big(\big(-\frac{1}{2}\big)$  $\Big) = -x^4\,\frac{(N^2-1)}{16N^2}$,
- $W_{su(N)}\Big($  $\Big) = x^4\,\frac{(N^2-1)^2}{16N^4}$,
- $W_{su(N)}\Big($  $\Big) = x^4\,\frac{(N^2-1)}{16N^4}$.